Further repairs, attempted closures, and the remaining estimate

An updated second companion to *The Music of the Primes*

Analysis of the uploaded 940-page draft by Alessandro Maria Masullo

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Status: proved component results, conditional consequences, and unsuccessful attempts are distinguished below. No proof of RH or GRH has been obtained.

This revision preserves the earlier operator and cutoff repairs and adds the subsequent arithmetic attempts. It gives exact identities and criteria for the missing estimate, explains the limitations of the attempted closures, and states the remaining task without assuming the needed cancellation.

Part	Result and limitation
Earlier analytic repairs	Trace-class condensation, local cutoff correction, Hardy-space closure, and positivity obstructions are retained in Sections 1–4. The full arithmetic comparison remains unproved.
Mollifier norm	Its power-growth exponent is identified with the zero-free boundary. Subpower growth is equivalent to RH.
Prime energy	The averaged discrepancy criterion, jump-energy identity, and one-sided covariance criterion are proved. The required bound is not.
Arithmetic inversion	Exact factorial forcing, floor terms, Möbius inverse, and gcd energy decomposition are proved. The signed aggregate remains.
Attempted closures	Smoothing, sign shortcuts, Selberg symmetry, shifted completed-function comparisons, and Gram estimates are audited.

The central target. With $\psi(x) = \sum_{n \le x} \Lambda(n)$ and $d(x) = \psi(2x)/(2x) - \psi(x)/x - \mathbf{1}_{[1,2)}(x)$, the bound called "(4)" in the discussion is

$$\int_1^X |d(x)|^2 \, dx \ll_\varepsilon X^\varepsilon \quad \text{for every } \varepsilon > 0.$$

It is restated as (61) and proved equivalent to RH in Theorem 6.1. This equivalence identifies the gap; it does not prove the bound.

Scope and inputs. The new detailed arithmetic calculations concern the Riemann zeta function. Section 9 states the separate all-character requirement for GRH. Consequences of the reported external 7/8 zero-free boundary are explicitly conditional on that input; its full proof and formal verification have not been independently audited here. The source manuscript and the preceding companion are unchanged.

Earlier qualification retained. Chapter 13's finite zero-moment conditions remove the untwisted constant shell defect in the rational trivial-character case. The remaining boundary and omitted-zero obstructions survive that correction. This supplement is not a line-by-line audit of the entire manuscript.

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Conventions. $\mathcal{S}_1$ denotes trace-class operators. Logarithmic convolution is $A_bF(x) = \int b(t)F(x-t)\,dt$. The polynomial norm is $H_s = L^2(\mathbb{R}, (1+x^2)^{s/2}\,dx)$. Additive Fourier transforms in the real-place source use $e^{-2\pi iu\xi}$; Fourier transforms in the logarithmic coordinate use $e^{-ix\xi}$. Every Mellin or Laplace convention needed for a calculation is stated locally. The letter s denotes a complex Laplace variable in the Hardy subsection and a real weight index where it occurs as a subscript on H_s. The function $\xi_\zeta(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$ is the completed entire zeta function.

1 A proved repair of the final Paley–Wiener condensation trace

This section supplies one of the missing operator estimates in the rational, critical-line, polynomially weighted model. It proves that the off-diagonal term dropped in the manuscript is trace class and that its localized trace tends to zero. The fixed-cutoff operator identity remains incorrect. The result does not identify the global arithmetic residual.

1.1 A general trace-class theorem for a real spectral jet space

For an integer $m \geq 1$ put

$$H_{2m} = L^2(\mathbb{R}, (1+x^2)^m\,dx), \qquad \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx.$$

Fourier transformation identifies this space, up to a constant factor, with $H^m(\mathbb{R})$ with inner product

$$\langle F, G\rangle_m = \sum_{j=0}^{m} \binom{m}{j} \int_{\mathbb{R}} F^{(j)}(\xi)\overline{G^{(j)}(\xi)}\,d\xi.$$

Its distributional Riesz operator is $L_m = (1-\partial_\xi^2)^m$.

Let $Z \subset \mathbb{R}$ be discrete, let $1 \leq r_z \leq m$ for $z \in Z$, and assume its local counting function has polynomial growth:

$$\#(Z \cap [k-3, k+3]) \leq C(1+|k|)^d, \qquad k \in \mathbb{Z}.$$

Define the closed subspace

$$M = \mathcal{F}^{-1}\{F \in H^m : F^{(j)}(z) = 0,\ z \in Z,\ 0 \leq j < r_z\}.$$

Let Q be its orthogonal projection and $E = I - Q$. Convolution $A_bf = b * f$, $b \in C_c^\infty(\mathbb{R})$, preserves M.

Theorem 1.1 (Trace-class off-diagonal term)**.** *The operators A_bE and EA_b are trace class. In particular*

$$B_b := QA_bE \in \mathcal{S}_1(H_{2m}), \qquad \operatorname{Tr} B_b = 0.$$

If ρ_R are uniformly bounded measurable functions tending to one almost everywhere, then

$$\|M_{\rho_R}B_b - B_b\|_1 \longrightarrow 0, \qquad \operatorname{Tr}(M_{\rho_R}B_b) \longrightarrow 0.$$

Proof. Work first on the Fourier side. If $v \in M^\perp$, then L_mv is supported on Z: test the orthogonality against smooth functions compactly supported off Z. Since $L_mv \in H^{-m}$, its restriction to a bounded interval is a finite sum of delta derivatives of orders at most $m-1$. On an interval J meeting q points of Z, the restrictions of all such v therefore lie in a vector space of dimension at most $2m + mq$. Indeed

choose particular distributional solutions for the at most mq point sources; the remaining solutions of $L_m v = 0$ form a $2m$-dimensional space. No separation bound on the points of Z is used.

Choose a smooth partition of unity $\{\chi_k\}_{k\in\mathbb{Z}}$ with uniformly bounded derivatives and supports in $[k-1, k+1]$, and put $c_k = \chi_k \widehat{b}$. The range of the multiplier $M_{c_k}E$ has dimension at most $2m + m\#(Z \cap [k-2, k+2])$. The Sobolev multiplier estimate gives

$$\|M_{c_k}E\|_1 \le C_m(1 + \#(Z \cap [k-2, k+2])) \sum_{j=0}^{m} \|c_k^{(j)}\|_\infty.$$

The right side is summable in k, because $\widehat{b}$ is Schwartz and the local zero count grows polynomially. The multiplier $M_{\widehat{b}}E$ is consequently a trace-norm sum of finite-rank operators. This proves $A_b E \in \mathcal{S}_1$. Invariance of M gives $EA_bQ = 0$, and hence $EA_b = EA_bE \in \mathcal{S}_1$. Also $B_b = Q(A_bE)$ is trace class and

$$\operatorname{Tr} B_b = \operatorname{Tr}((A_bE)Q) = 0.$$

Finally $M_{\rho_R} \to I$ strongly, with a uniform operator norm bound. Finite-rank approximation of B_b proves the trace-norm limit. □

The same finite-frequency multiplier argument computes the diagonal block:

$$\operatorname{Tr}(EA_b) = \sum_{z\in Z} r_z \widehat{b}(z).$$

For a compactly supported frequency multiplier, the induced quotient map has finite rank. In its jet coordinates it is triangular, with diagonal value $c(z)$ repeated r_z times. Summing the partition of unity in trace norm proves the formula, which converges absolutely by the local count bound. Thus both the off-diagonal estimate and the spectral trace are obtained without assuming a well-conditioned eigenvector basis.

1.2 An independent identification of the rational critical-line space

Set

$$f_0(u) = (3u^2 - 2\pi u^4)e^{-\pi u^2}, \qquad K(x) = \sum_{n\neq 0} f_0(ne^x), \qquad K_{1/2}(x) = e^{x/2}K(x).$$

Both $f_0(0)$ and $\widehat{f_0}(0)$ vanish. Poisson summation implies that K and all its derivatives decay faster than every exponential at both ends. The exact Mellin calculation is

$$\widehat{K_{1/2}}(\xi) = -\frac{\xi_\zeta(1/2 - i\xi)}{\pi}, \qquad \xi_\zeta(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Let

$$Z = \{\xi \in \mathbb{R} : \zeta(1/2 - i\xi) = 0\}, \qquad r_\xi = \min\{m, \operatorname{ord}_{1/2-i\xi}\zeta\}.$$

The classical zero-counting bound implies the local polynomial bound required in Theorem 1.1.

Proposition 1.2 (Cyclic theta completion)**.** *The closed span of the translates of $K_{1/2}$ in H_{2m} is precisely the jet-kernel space M above. It also equals the closure of the half-twisted rational theta profiles*

$$e^{x/2}\sum_{n\neq 0} f(ne^x), \qquad f \in \mathcal{S}(\mathbb{R}) \text{ even}, \ f(0) = \widehat{f}(0) = 0.$$

No statement about zeros off the critical line is assumed.

Proof. An annihilator of the translates defines a tempered distribution whose Fourier transform is killed by $\widehat{K_{1/2}}$. Locally it is a combination of delta derivatives at its real zeros, of order below the corresponding multiplicity. Membership in H^{-m} permits exactly orders $0, \dots, m-1$. Expanding smooth cutoffs approximate such distributions in H^{-m}, so these local jets generate the full annihilator. Taking annihilators gives the asserted closure.

Every other displayed theta profile has the same required vanishing jets. Initially on $\operatorname{Re} s > 1$ its Mellin transform is

$$2\zeta(s) \int_0^\infty f(u) u^{s-1}\, du.$$

At nontrivial zeros on the critical line the second factor is holomorphic: $f(0) = 0$ and evenness give $f(u) = O(u^2)$ near zero, and Schwartz decay controls infinity. Poisson summation and $\widehat{f}(0) = 0$ justify the entire theta Mellin transform and continuation of the identity to those points. Thus every such profile belongs to M. Conversely the family contains all translates of $K_{1/2}$, up to harmless nonzero scalars from dilation. □

1.3 The actual rational global Paley–Wiener exhaustion

For $\Omega > 0$ let M_Ω be the H_{2m} closure of the same half-twisted theta profiles with the additional condition

$$\operatorname{supp} \widehat{f} \subset [-\Omega, \Omega].$$

This is the rational global specialization of the manuscript's Paley–Wiener family: it first takes the local even function, then globalizes it by $f \mapsto \sum_{n \neq 0} f(ne^x)$, and only then takes the half-twist and Hilbert closure. It must be distinguished from the independent local bandlimited projector on additive $L^2(\mathbb{R})$. The relevant definitions are 7.3.2.1, pp. 351–352, and 11.2.1.1, pp. 487–488.

Let P_Ω be the orthogonal projection onto M_Ω.

Proposition 1.3 (Strong exhaustion)**.** *The spaces M_Ω increase, and $P_\Omega \to Q$ strongly as $\Omega \to \infty$.*

Proof. Choose even smooth frequency cutoffs χ_Ω equal to one near zero and exhausting the line. For even $f \in \mathcal{S}$ with the two vanishing conditions, $\chi_\Omega \widehat{f} \to \widehat{f}$ in Schwartz topology and retains the value zero at the origin. Its integral need not be zero. Correct that integral by subtracting $c_\Omega \eta$, where η is a fixed even smooth frequency bump supported away from zero with integral one, and $c_\Omega = \int \chi_\Omega \widehat{f} \to 0$. The corrected transforms are compactly supported, vanish at zero, have integral zero, and converge in Schwartz topology to $\widehat{f}$. Their inverse transforms therefore have both required conditions.

The theta map on this subspace is continuous from Schwartz topology to H_{2m} after the half-twist. At large positive x use the Schwartz tails of $f(ne^x)$. At large negative x use Poisson summation and the Schwartz tails of $\widehat{f}(ne^{-x})$; the zero terms vanish. Arbitrarily high Schwartz orders dominate the fixed polynomial logarithmic weight. Thus the corrected bandlimited profiles converge in H_{2m} to the original profile. The preceding proposition identifies the closure of their union as M. Monotone convergence of orthogonal projections proves the result. □

1.4 The corrected condensation trace

Let $b \in C_c^\infty(\mathbb{R})$ with $b(0) = 0$, and take smooth compact cutoffs ρ_R, uniformly bounded and converging pointwise to one. The localized operator

$$T_R = M_{\rho_R} Q A_b$$

is trace class: $M_{\rho_R}A_b$ has a smooth compactly supported kernel, and $M_{\rho_R}EA_b$ is trace class by Theorem 1.1. For each fixed R, strong exhaustion yields the *correct* limit

$$T_R P_\Omega \longrightarrow T_R Q \quad \text{in trace norm.}$$

The discarded term is

$$T_R - T_R Q = M_{\rho_R} Q A_b E.$$

It need not be zero at a fixed cutoff. The theorem above proves that its trace tends to zero when $R \to \infty$. Since $\operatorname{Tr}(M_{\rho_R}A_b) = b(0)\int \rho_R = 0$, we obtain the fully justified iterated limit

$$\begin{aligned}\lim_{R\to\infty}\lim_{\Omega\to\infty} \operatorname{Tr}(M_{\rho_R} Q A_b P_\Omega) &= -\operatorname{Tr}(EA_b) \\ &= -\sum_{\zeta(1/2-i\xi)=0} \min\{m, \operatorname{ord}_{1/2-i\xi}\zeta\}\,\widehat{b}(\xi).\end{aligned}$$

Increasing m restores the full multiplicities in this spectral sum by dominated convergence and the unconditional zero-counting bound.

This is a genuine repair of the final condensation trace for the specified rational polynomial completion. The Riesz-operator proof uses the local differential operator $(1-\partial_\xi^2)^m$; it has not established an analogous estimate for an exponential-width norm. Most importantly, it has not identified this global Paley–Wiener trace with the independently computed local arithmetic trace. The spectral content proved here remains the critical-line spectrum selected by the ambient topology.

2 A support-sensitive real-place comparison and its boundary term

This section gives an explicit comparison with both spatial and additive Fourier cutoffs in the case $F = \mathbb{Q}$, $S = \{\infty\}$, and the trivial norm-one character. It proves a local trace formula with its lower-boundary correction, including the Paley–Wiener condition $f(0) = \widehat{f}(0) = 0$. It also determines how the support bound changes during arithmetic depletion. These calculations do not identify the resulting local operator with the manuscript's global geometric projector.

2.1 What strong Paley–Wiener exhaustion actually proves

Let P be an orthogonal projector on a Hilbert space, let P_j be orthogonal projectors with $P_j \to P$ strongly, and suppose that $T_\rho = M_\rho P A$ is trace class. The elementary trace-ideal convergence statement is

$$\|T_\rho P_j - T_\rho P\|_1 \longrightarrow 0. \tag{1}$$

Indeed, finite-rank operators are dense in trace norm, and right multiplication by the uniformly bounded, strongly converging self-adjoint operators $P_j - P$ converges in trace norm on every finite-rank operator. If $A \operatorname{Ran} P \subset \operatorname{Ran} P$, then

$$T_\rho P = M_\rho A P, \qquad T_\rho - T_\rho P = M_\rho P A (I - P).$$

Thus the right side of the operator limit in manuscript Theorem 11.2.1.2(2), printed p. 488, needs a final P^{geom} unless an additional condition removes the second term. This is not merely a question of how an exhaustion is chosen: manuscript Remark 10.4.1.4, printed pp. 457–458, explicitly distinguishes $PA(I-P)$ from zero.

For the final, spatially exhausted *trace*, a weaker repair suffices. If $B = PA(I-P)$ is trace class and $M_\rho \to I$ strongly with uniformly bounded norms, then

$$\operatorname{Tr}(M_\rho B) \longrightarrow \operatorname{Tr} B = 0,$$

the last equality following from cyclicity and $(I-P)P = 0$. Theorem 1.1 proves this trace-class condition in the polynomial-weight, critical-line model. It repairs the final trace limit in that model, but does not justify the fixed-cutoff operator identity or a uniform interchange with an arithmetic exhaustion.

2.2 The local Hilbert space and its actual Paley–Wiener projector

Use the additive Fourier convention

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-2\pi i u\xi}\, du, \qquad \mathcal{H} = L^2_{\text{even}}(\mathbb{R}, du).$$

The map

$$(Vf)(x) = \sqrt{2}\, e^{x/2} f(e^x) \tag{2}$$

is unitary from $\mathcal{H}$ onto $L^2(\mathbb{R}, dx)$. The unitary dilations

$$U_t f(u) = e^{-t/2} f(e^{-t} u)$$

become translations $(VU_tV^{-1}F)(x) = F(x-t)$. Consequently, for a positive-norm test ϕ put

$$b(t) = \phi(e^t), \qquad A_b = \int_{\mathbb{R}} b(t) U_t\, dt.$$

The condition in manuscript (13.3.1.2) for the rational trivial isotype is exactly $\int b(t)\, dt = 0$ in this normalization. Throughout the trace calculation below assume $b \in C_c^\infty((0,\infty))$. In particular, the singular point of the local principal value is absent.

Let

$$B_\Omega = \mathcal{F}^{-1} 1_{[-\Omega,\Omega]} \mathcal{F}, \qquad k_\Omega(u) = \frac{\sin(2\pi\Omega u)}{\pi u}, \qquad q_\Omega = (2\Omega)^{-1/2} k_\Omega.$$

At $u = 0$ the displayed kernel is interpreted by continuity. We have $\|q_\Omega\|_2 = 1$, and q_Ω is even.

Lemma 2.1 (The zero-value Paley–Wiener projector)**.** *The L^2 closure of*

$$\{f \in \mathcal{S}(\mathbb{R}) : f \text{ even},\ \operatorname{supp} \widehat{f} \subset [-\Omega, \Omega],\ f(0) = \widehat{f}(0) = 0\}$$

is

$$\mathcal{H}^0_\Omega = \{f \in B_\Omega \mathcal{H} : f(0) = 0\}.$$

Its orthogonal projector is

$$B^0_\Omega = B_\Omega - |q_\Omega\rangle\langle q_\Omega|. \tag{3}$$

As $\Omega \to \infty$, $B^0_\Omega \to I$ strongly on $\mathcal{H}$.

Proof. On the bandlimited subspace, evaluation at zero is the bounded functional $f(0) = \int_{-\Omega}^{\Omega} \widehat{f}(\xi)\, d\xi$, with Riesz vector k_Ω and squared norm 2Ω. This gives the claimed projector. In Fourier coordinates, even smooth functions compactly supported in $(-\Omega, \Omega)$ and with integral zero are dense in the corresponding zero-integral L^2 hyperplane. The extra point condition $\widehat{f}(0) = 0$ does not change this closure: remove its value at zero with an arbitrarily narrow smooth even bump, and correct the resulting

arbitrarily small integral with a fixed bump supported away from zero. The two corrections can have arbitrarily small L^2 norm.

Finally, $B_\Omega \to I$ strongly. The normalized Fourier transforms $\widehat{q}_\Omega = (2\Omega)^{-1/2} 1_{[-\Omega,\Omega]}$ tend weakly to zero in L^2, first against compactly supported functions and then by density. Thus the rank-one projectors in (3) tend strongly to zero. □

This is an exact projector calculation for the unweighted local half-twisted model. It is not a formula for the orthogonal projector onto the global theta image in a weighted Sobolev completion. That latter identification would require a separate theorem.

There is also a useful distinction between $\mathcal{S}_0$ and $\mathcal{S}_{00}$. Manuscript Theorem 5.3.2.1, printed p. 243, defines $\mathcal{S}_{00}(\mathbb{R})$ by vanishing of every derivative at zero. A bandlimited Schwartz function extends to an entire function; if all of its derivatives at zero vanish, it is zero. Hence a nonzero additive bandlimited function cannot simultaneously lie in $\mathcal{S}_{00}$. A smooth spatial cutoff vanishing near zero can restore membership in $\mathcal{S}_{00}$ *after* the Fourier projection, but the upper-only cutoff in the bullet on printed p. 516 does not have this property. The lower cutoff proposed in the footnote on printed p. 514 changes the trace by the term now computed.

2.3 An exact two-cutoff trace formula

For $0 \le \epsilon < R$ write

$$C_{\epsilon,R} = 1_{\{\epsilon \le |u| \le R\}}, \qquad L = R\Omega, \quad \ell = \epsilon\Omega.$$

We use a sandwich to obtain an unambiguous trace-class operator:

$$T_{\epsilon,R,\Omega}(b) = C_{\epsilon,R} A_b B_\Omega C_{\epsilon,R}, \qquad T^0_{\epsilon,R,\Omega}(b) = C_{\epsilon,R} A_b B^0_\Omega C_{\epsilon,R}.$$

Since the spatial cutoff is a projector, cyclicity identifies these traces with the corresponding compressed traces on the bandlimited subspaces. This formulation does not assume that an unsandwiched operator is trace class.

Define $\mathrm{Si}(z) = \int_0^z (\sin v)/v \, dv$ and, for $c > 0$,

$$I_z(c) = \frac{2}{\pi c} \mathrm{Si}(2\pi c z), \qquad I_z(0) = 4z. \tag{4}$$

Also put $k(v) = \sin(2\pi v)/(\pi v)$, and for $t > 0$ set

$$H_z(t) = \frac{e^{-t/2}}{2} [I_z(1 - e^{-t}) + I_z(1 + e^{-t})], \tag{5}$$

$$J_z(t) = \frac{e^{-t/2}}{2} \int_{|v| \le z} k(e^{-t} v) k(v) \, dv, \tag{6}$$

$$K_\infty(t) = \frac{e^{-t/2}}{1 - e^{-2t}} = \frac{1}{4 \sinh(t/2)} + \frac{1}{4 \cosh(t/2)}. \tag{7}$$

Theorem 2.2 (Exact local comparison, including the lower boundary)**.** *For $b \in C_c^\infty((0,\infty))$ both operators above are trace class on $\mathcal{H}$, and*

$$\mathrm{Tr}\, T_{\epsilon,R,\Omega}(b) = \int_0^\infty b(t) \, [H_L(t) - H_\ell(t)] \, dt, \tag{8}$$

$$\mathrm{Tr}\, T^0_{\epsilon,R,\Omega}(b) = \int_0^\infty b(t) [H_L(t) - J_L(t) - H_\ell(t) + J_\ell(t)] \, dt. \tag{9}$$

As $L \to \infty$, uniformly for t in compact subsets of $(0,\infty)$,

$$H_L(t) \longrightarrow K_\infty(t), \qquad J_L(t) \longrightarrow e^{-t/2}. \tag{10}$$

Consequently, if $L \to \infty$ and $\ell \to \tau \in [0,\infty)$, then

$$\operatorname{Tr} T_{\epsilon,R,\Omega}(b) \longrightarrow \int b(t)[K_\infty(t) - H_\tau(t)]\,dt, \tag{11}$$

$$\operatorname{Tr} T^0_{\epsilon,R,\Omega}(b) \longrightarrow \int b(t)[K_\infty(t) - e^{-t/2} - H_\tau(t) + J_\tau(t)]\,dt. \tag{12}$$

If instead $L, \ell \to \infty$, both traces tend to zero.

Proof. On full additive L^2, the operators $C_{\epsilon,R}U_tB_\Omega$ and $B_\Omega C_{\epsilon,R}$ are Hilbert–Schmidt, with squared norms $4\Omega(R-\epsilon)e^{-t}$ and $4\Omega(R-\epsilon)$ respectively. Their product is $C_{\epsilon,R}U_tB_\Omega C_{\epsilon,R}$, so its trace exists. Integrating in t over the compact support of b is permitted in trace norm. The rank-one correction in (3) preserves trace class.

The full-space diagonal calculation gives

$$\operatorname{Tr}(C_{\epsilon,R}U_tB_\Omega C_{\epsilon,R}) = e^{-t/2}\int_{\epsilon\le|u|\le R}\int_{-\Omega}^{\Omega} e^{2\pi i(e^{-t}-1)u\xi}\,d\xi\,du.$$

All the integrals here have finite domains. The trace identity also follows directly by the Hilbert–Schmidt product formula, so no unproved diagonal theorem is needed. For reflection $(Jf)(u) = f(-u)$, the trace with J inserted replaces $e^{-t}-1$ by $-e^{-t}-1$. Since all the operators commute with J, their trace on the even subspace is one half the sum of these two full traces. Finally,

$$\int_{-R}^{R}\int_{-\Omega}^{\Omega} e^{2\pi icu\xi}\,d\xi\,du = I_{R\Omega}(|c|).$$

Subtracting the inner interval gives (8).

The rank-one trace to subtract is

$$\langle C_{\epsilon,R}U_tq_\Omega, C_{\epsilon,R}q_\Omega\rangle = \frac{e^{-t/2}}{2}\int_{\ell\le|v|\le L} k(e^{-t}v)k(v)\,dv = J_L(t) - J_\ell(t).$$

This proves (9).

As $z \to \infty$, $I_z(c) \to c^{-1}$, uniformly when c stays in a compact subset of $(0,\infty)$. This gives the limit for H_L. For $r = e^{-t} < 1$, Parseval's formula gives

$$\int_{\mathbb{R}} k(rv)k(v)\,dv = \int_{\mathbb{R}} r^{-1}1_{[-r,r]}(\xi)1_{[-1,1]}(\xi)\,d\xi = 2.$$

The bound $|k(v)| \le \min\{2, (\pi|v|)^{-1}\}$ controls its tails uniformly on compact positive t-intervals, and gives the limit for J_L. The remaining assertions follow by integration against b. □

Corollary 2.3 (A proved sufficient coupling)**.** *If*

$$R\Omega \longrightarrow \infty, \qquad \epsilon\Omega \longrightarrow 0, \tag{13}$$

then

$$\operatorname{Tr} T_{\epsilon,R,\Omega}(b) \longrightarrow W_\infty(b),$$
$$\operatorname{Tr} T^0_{\epsilon,R,\Omega}(b) \longrightarrow W_\infty(b) - M_-(b),$$

where

$$W_\infty(b) = \int b(t)K_\infty(t)\,dt, \qquad M_-(b) = \int b(t)e^{-t/2}\,dt.$$

For b supported in a fixed compact subinterval of $(0,\infty)$, the errors are bounded by

$$C\|b\|_{L^1}\big((R\Omega)^{-1} + \epsilon\Omega\big).$$

In the parameters of the footnote on printed p. 514, $\epsilon = 1/\lambda''$, $\Omega = \lambda'$, so a sufficient condition is $\lambda'/\lambda'' \to 0$, together with $\lambda\lambda' \to \infty$.

Proof. Integration by parts in the sine-integral tail bounds $|I_L(c) - c^{-1}|$ by $C/(Lc^2)$. Also $|I_\ell(c)| \le 4\ell$. The bound on k used above gives $|J_L(t) - e^{-t/2}| \le C/L$ and $|J_\ell(t)| \le C\ell$, uniformly on the stated compact interval. □

The factor $\frac{1}{2}$ from the even subspace is exactly the factor in the real-place term of manuscript Theorem 12.4.1.1: writing $g = \pm e^t$ in that term gives

$$\frac{1}{2}\int_{\mathbb{R}^\times} \frac{\phi(|g|)|g|^{1/2}}{|1-g|}\,\frac{dg}{|g|} = W_\infty(b)$$

when the test is supported on positive logarithmic norm. Thus (13) supplies an actual local comparison, with the lower-boundary and zero-value corrections displayed explicitly.

2.4 The boundary discrepancy survives the stated moment condition

At any fixed $0 < \epsilon < R$, Theorem 2.2 gives

$$\lim_{\Omega\to\infty} \operatorname{Tr} T^0_{\epsilon,R,\Omega}(b) = 0. \tag{14}$$

Subsequently sending $\epsilon \downarrow 0$ and $R \to \infty$ still gives zero. The coupled limit (13) instead gives $W_\infty(b) - M_-(b)$. In particular, increasing the Paley–Wiener level first at a fixed spatial annulus is not interchangeable with the local trace limit.

This discrepancy is not removed by imposing $\int b = 0$. It persists even if both polar moments also vanish. To see this without a genericity argument, choose a nonzero $a \in C_c^\infty((0,\infty))$ with $a \ge 0$ and put

$$b = \frac{d}{dt}\left(\frac{d^2}{dt^2} - \frac{1}{4}\right)a. \tag{15}$$

Integration by parts proves

$$\int b = \int b(t)e^{t/2}\,dt = \int b(t)e^{-t/2}\,dt = 0.$$

For $t > 0$ the locally normally convergent expansion

$$K_\infty(t) = \sum_{n=0}^{\infty} e^{-(2n+1/2)t}$$

and another integration by parts give

$$W_\infty(b) = \int a(t)\sum_{n=1}^{\infty}(2n+\tfrac{1}{2})((2n+\tfrac{1}{2})^2 - \tfrac{1}{4})e^{-(2n+1/2)t}\,dt > 0. \tag{16}$$

For this explicit class of tests, the two limits are zero and a strictly positive number, even on the moment-restricted space of Chapter 13 and after the local rank-one correction. This is a counterexample to the unjustified interchange of these local cutoff limits; it is not a counterexample to GRH or, by itself, a disproof of a correctly formulated global comparison with an additional boundary term.

For orientation, the earlier untwisted polynomial shell defect $-\int b$ does vanish on this restricted test space. If a quotient is transported by $U_\beta F(x) = e^{\beta x}F(x)$, however, the action of A_b on the transported quotient corresponds to $A_{e^{-\beta t}b}$, since

$$U_\beta^{-1} A_b U_\beta = A_{e^{-\beta t}b}.$$

The transported common-zero defect would therefore be $-\int e^{-\beta t}b(t)\,dt$, which is a different condition. The explicit test (15) kills both defects when $\beta = 1/2$, so neither explains away the boundary discrepancy just proved.

2.5 The support information during arithmetic depletion

The following calculation uses actual bandlimited Schwartz representatives, not arbitrary elements of an enlarged completed range. Fix $X > 0$, choose an even real $g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset [-X, X]$ and $g(0) = \int g = 0$, and put $f = \mathcal{F}^{-1}g$. Then f is an even Schwartz function satisfying $f(0) = \widehat{f}(0) = 0$. Define

$$\Theta_f(t) = \sum_{n \in \mathbb{Z}\setminus\{0\}} f(nt), \qquad t > 0.$$

If $p_1, \dots, p_N$ are distinct primes, put $M_N = \prod_{j=1}^N p_j$ and

$$\Theta_{f,N}(t) = \prod_{j=1}^N (I - T_{\log p_j})\Theta_f(t) = \sum_{d \mid M_N} \mu(d)\Theta_f(dt) = \sum_{\substack{n \neq 0 \\ (n, M_N)=1}} f(nt). \tag{17}$$

Here T_h is translation by h in logarithmic coordinates, equivalently the substitution $t \mapsto e^h t$ before those coordinates are taken.

Proposition 2.4 (The arithmetic support bound is not uniform)**.** *Poisson summation gives*

$$\Theta_f(t) = t^{-1} \sum_{n \neq 0} g(n/t),$$

so $\Theta_f(t) = 0$ *for* $t < 1/X$ *and*

$$\Theta_{f,N}(t) = 0 \quad \text{for } t < 1/(XM_N). \tag{18}$$

If the positive endpoint of $\operatorname{supp} g$ *is* X*, this lower support bound is sharp. Multiplication by* t^β *for any real* β *preserves it.*

Proof. The first assertion is Poisson summation with the zero terms removed, using $f(0) = g(0) = 0$. The finite inclusion–exclusion sum in (17) then proves (18). For $1/(XM_N) < t < 2/(XM_N)$, every proper divisor d of M_N satisfies $dt < 1/X$, whereas the M_N term equals

$$\mu(M_N)\Theta_f(M_N t) = \frac{2\mu(M_N)}{M_N t}\, g\left(\frac{1}{M_N t}\right).$$

If $\operatorname{supp} g$ reaches X, this expression is nonzero arbitrarily close to $1/(XM_N)$ from above. The half-twist is nonvanishing on $t > 0$ and does not change support. □

Thus a fixed Archimedean Fourier bandwidth does not give a uniform lower global-norm support bound during depletion of more primes. In logarithmic coordinates the lower bound moves to $-\log X - \log M_N$. This is compatible with a fixed-S' version of Proposition 7.3.2.2; it prevents treating its support constant as independent of the expanding finite set $S' \setminus S$. The missing dependence has an elementary Fourier explanation:

$$\mathcal{F}_{\mathbb{Q}_p} 1_{\mathbb{Z}_p^\times} = 1_{\mathbb{Z}_p} - p^{-1} 1_{p^{-1}\mathbb{Z}_p}$$

for the standard self-dual measure and character. Replacing more unit-ball factors by unit-group factors enlarges the non-Archimedean Fourier support, although the Archimedean factor g remains fixed.

2.6 An escaping boundary term with fixed Archimedean bandwidth

The support drift has a quantitative consequence in the very narrow-strip norms for which the half-twisted finite factors are invertible. Let the primes in (17) be the first N primes, and let $Y_N = p_{N+1}$. For $0 < \beta < 1$ write

$$G_N(x) = e^{\beta x}\Theta_{f,N}(e^x), \qquad G_\infty(x) = 2e^{\beta x} f(e^x).$$

Let

$$H_{a,s} = L^2\big(\mathbb{R}, e^{2a|x|}(1+x^2)^{s/2}\,dx\big), \qquad a > 0, \quad s \geq 0.$$

Each G_N belongs to every such space, since it has a lower support bound and rapid upper decay. If $a < \beta + 2$, then $G_\infty \in H_{a,s}$ as well, since the even function f has $f(0) = 0$.

Choose g so that its Mellin transform after Fourier inversion is nonzero on $0 < z < 1$. Such a choice is elementary: on the positive axis take a normalized nonnegative bump on an inner interval minus a normalized nonnegative bump on a disjoint outer interval, both away from zero, and extend evenly. For $0 < z < 1$, Abel regularization of the cosine transform gives

$$\mathcal{M}f(z) := \int_0^\infty f(t) t^{z-1}\,dt = 2\Gamma(z)\cos(\pi z/2)(2\pi)^{-z}\int_0^\infty g(\xi)\xi^{-z}\,d\xi > 0. \tag{19}$$

The inequality follows from strict decrease of ξ^{-z}.

Proposition 2.5 (A nonzero local limit with divergent global norms)**.** *For this fixed bandlimited representative and every fixed lower cutoff $L_0 \in \mathbb{R}$,*

$$1_{[L_0,\infty)} G_N \longrightarrow 1_{[L_0,\infty)} G_\infty \quad \text{in } H_{a,s}.$$

Nevertheless,

$$\int_{\mathbb{R}} G_N(x)\,dx = 2\zeta(\beta)\mathcal{M}f(\beta)\prod_{j=1}^{N}(1 - p_j^{-\beta}) \longrightarrow 0, \tag{20}$$

$$\int_{\mathbb{R}} G_\infty(x)\,dx = 2\mathcal{M}f(\beta) > 0. \tag{21}$$

For every $0 < \delta < 1$ there are constants $c_\delta > 0$ and N_δ such that

$$\|G_N\|_{H_{a,s}} \geq c_\delta Y_N^{a\delta}(\log Y_N)^{s/2} \qquad (N \geq N_\delta). \tag{22}$$

Proof. All surviving positive integers except 1 are at least Y_N. For any $Q > 1$, the Schwartz bound gives, whenever $Y_N t \geq 1$,

$$|\Theta_{f,N}(t) - 2f(t)| \leq 2\sum_{n \geq Y_N} |f(nt)| \leq C_Q Y_N^{1-Q} t^{-Q}.$$

This proves the stated convergence on every fixed lower half-line after multiplication by t^β and integration against the weight. More precisely, on $x \geq -\delta \log Y_N$ its unweighted L^1 norm is at most

$$C_Q Y_N^{1-\delta\beta-(1-\delta)Q},$$

which tends to zero for sufficiently large Q. The weighted L^2 version follows similarly, increasing Q if needed.

For $\Re z > 1$, termwise Mellin integration gives $\mathcal{M}\Theta_f(z) = 2\zeta(z)\mathcal{M}f(z)$. The left side is entire because of its lower support bound and rapid upper decay. On $\Re z > 0$ the only apparent pole on the right is cancelled by $\mathcal{M}f(1) = \widehat{f}(0)/2 = 0$. Analytic continuation and finite inclusion–exclusion give (20). The Euler product tends to zero: for $0 < \beta < 1$ its factors are no larger than $1 - p^{-1}$, and

$$\prod_{p \leq Y} (1 - p^{-1})^{-1} \geq \sum_{n \leq Y} \frac{1}{n} \longrightarrow \infty.$$

No assertion about zeros of ζ is used.

The preceding L^1 estimate implies

$$\int_{x \geq -\delta \log Y_N} G_N(x)\, dx \longrightarrow 2\mathcal{M}f(\beta).$$

Together with (20), this shows that the integral on $x < -\delta \log Y_N$ tends to $-2\mathcal{M}f(\beta) \neq 0$. For $L > 0$, Cauchy–Schwarz bounds this lower-tail integral by

$$\|G_N\|_{H_{a,s}} \left(\int_L^\infty e^{-2au}(1+u^2)^{-s/2}\, du \right)^{1/2} \leq \frac{\|G_N\|_{H_{a,s}}}{\sqrt{2a}} e^{-aL}(1+L^2)^{-s/4}.$$

Taking $L = \delta \log Y_N$ proves (22). □

The preceding scalar witness is a zeroth moment, so convolution with a test satisfying $\int b = 0$ kills that particular witness. The underlying failure of bounded global convergence is stronger and survives every nonzero compactly supported convolution test.

Corollary 2.6 (Moment-restricted tests do not restore boundedness)**.** *Let $0 < a < \beta + 2$, $s \geq 0$, and let $0 \neq b \in C_c^\infty(\mathbb{R})$, with any prescribed finite set of vanishing moments. Then*

$$\|A_b G_N\|_{H_{a,s}} \longrightarrow \infty,$$

where A_b is now convolution in the logarithmic variable. There is still convergence to $A_b G_\infty$ on every fixed lower half-line in $H_{a,s}$.

Proof. The entire function $B(z) = \int b(t) e^{zt}\, dt$ is not identically zero. Choose a real η with $|\eta| < a$, $0 < \beta + \eta < 1$, and $B(\eta) \neq 0$. Such a choice exists because the real interval is open and nonempty and the zeros of a nonzero entire function are isolated. The functional $\ell_\eta(F) = \int F(x) e^{\eta x}\, dx$ is continuous on $H_{a,s}$. The same calculation as in (20) gives

$$\ell_\eta(A_b G_N) = 2B(\eta)\zeta(\beta+\eta)\mathcal{M}f(\beta+\eta) \prod_{j=1}^N (1 - p_j^{-(\beta+\eta)}) \longrightarrow 0,$$

whereas

$$\ell_\eta(A_b G_\infty) = 2B(\eta)\mathcal{M}f(\beta+\eta) \neq 0.$$

Compact support of b and the preceding estimates give the asserted local convergence. If a subsequence of $A_b G_N$ were bounded in the Hilbert space, a further subsequence would converge weakly. Local convergence forces its weak limit to be $A_b G_\infty$, while continuity of ℓ_η contradicts the displayed limits. Thus no bounded subsequence exists, which is the claimed norm divergence. □

This result is valid in particular for $\beta = 1/2$ and $0 < a < 1/2$, and for the test (15). It does not prohibit a renormalized trace comparison: the mass that escapes to smaller norm is precisely information such a comparison must retain. It does rule out obtaining that comparison from bounded convergence of these depleted representatives in the same global Hilbert norm, even at fixed Archimedean Fourier bandwidth.

2.7 An exact Hardy-space description of the unweighted global level

The local operator B_Ω^0 and the global theta Paley–Wiener projector are different objects. Their distinction can be made exact without assuming RH. In this subsection only, take the unweighted completion $s = 0$ and half-twist $\beta = 1/2$. Let $\mathcal{W}_\Omega$ be the closure in $L^2(\mathbb{R}, dx)$ of

$$G_f(x) = e^{x/2} \sum_{n \neq 0} f(ne^x), \quad f \in \mathcal{S}(\mathbb{R}) \text{ even}, \quad f(0) = \widehat{f}(0) = 0, \quad \operatorname{supp} \widehat{f} \subset [-\Omega, \Omega]. \tag{23}$$

This is the rational, trivial-isotype, $S = \{\infty\}$, $S_0 = \varnothing$ specialization of the global level in Definition 11.2.1.1. The local theta function of even f is $2f(t)$, and its globalization is $\sum_{n\neq 0} f(nt)$; the definition then applies the half-twist and takes the indicated Hilbert closure.

By Proposition 2.4, $\mathcal{W}_\Omega \subset L^2([-\log \Omega, \infty))$. After translation of the lower endpoint to zero, all these spaces reduce to $\mathcal{W}_1$. Write

$$(\mathcal{L}h)(s) = \int_0^\infty h(x) e^{-sx}\, dx, \qquad \Re s > 0.$$

The Laplace transform identifies $L^2(0, \infty)$ with the Hardy space $H^2(\mathbb{C}_+)$, up to the fixed Fourier normalization of its norm.

Theorem 2.7 (The actual unweighted level retains a Blaschke factor)**.** *Set*

$$D(s) = \frac{(s - \frac{1}{2})\zeta(s + \frac{1}{2})}{(s+1)^2}, \qquad \Re s > 0, \tag{24}$$

with its removable value at $s = 1/2$. Then $D \in H^\infty(\mathbb{C}_+)$ and

$$\mathcal{L}\mathcal{W}_1 = \overline{DH^2(\mathbb{C}_+)}^{H^2} = B_\zeta H^2(\mathbb{C}_+), \tag{25}$$

where B_ζ is the Blaschke product whose zeros, with multiplicity, are $\rho - 1/2$ for the zeros ρ of ζ with $\Re\rho > 1/2$. There is no additional singular inner factor. Consequently,

$$\mathcal{W}_1 = L^2(0, \infty) \quad \Longleftrightarrow \quad \zeta(\rho) \neq 0 \text{ for } \Re\rho > 1/2 \quad \Longleftrightarrow \quad \text{RH}. \tag{26}$$

The last equivalence uses the usual functional-equation symmetry of the nontrivial zeros. No equality of weighted H_{2m} spaces is asserted in this theorem.

Proof. First take $\Omega = 1$, put $g = \widehat{f}$, and define

$$h(x) = 2e^{-x/2} g(e^{-x}), \qquad x \geq 0.$$

Because g is even and smooth with $g(0) = 0$, its value at $u = 0$ is $O(u^2)$. Thus h and all its derivatives decay at least as $e^{-5x/2}$ as $x \to \infty$. At the lower endpoint h is flat because g is smooth and supported in $[-1, 1]$. Moreover,

$$(\mathcal{L}h)(1/2) = 2 \int_0^1 g(u)\, du = 0.$$

Poisson summation gives the locally finite causal convolution

$$G_f(x) = \sum_{n\ge 1} n^{-1/2} h(x - \log n), \qquad h(y) = 0 \ (y < 0). \tag{27}$$

For $\Re s > 1/2$, absolute convergence permits its Laplace transform:

$$(\mathcal{L}G_f)(s) = \zeta(s + 1/2)(\mathcal{L}h)(s). \tag{28}$$

The pole at $s = 1/2$ is cancelled by the displayed moment of h.

Solve $k' - \frac{1}{2}k = h$ using

$$k(x) = e^{x/2} \int_0^x e^{-u/2} h(u)\, du = -e^{x/2} \int_x^\infty e^{-u/2} h(u)\, du.$$

The second equality is precisely the zero moment. Thus k, k', k'' are in $L^2(0,\infty)$, decay exponentially, and have zero boundary values needed for integration by parts. Let $r = (\partial_x + 1)^2 k \in L^2$. Then

$$\mathcal{L}h = (s - 1/2)\mathcal{L}k, \qquad \mathcal{L}r = (s + 1)^2 \mathcal{L}k, \qquad \mathcal{L}G_f = D\mathcal{L}r.$$

This proves the inclusion of the source core in DH^2 once boundedness of D is established below.

Conversely, take any $k \in C_c^\infty(0,\infty)$ and put $h = (\partial_x - 1/2)k$. Then $\mathcal{L}h(1/2) = 0$ and

$$g(u) = \tfrac{1}{2} u^{-1/2} h(-\log u), \qquad 0 < u < 1,$$

extended evenly and by zero elsewhere, is smooth, supported away from both 0 and the bandwidth edge, and satisfies $g(0) = \int g = 0$. It therefore gives an allowed source in (23). Its global image has Laplace transform $D\mathcal{L}[(\partial_x + 1)^2 k]$. The range

$$\{(\partial_x + 1)^2 k : k \in C_c^\infty(0,\infty)\}$$

is dense in $L^2(0,\infty)$. Indeed, an annihilator in L^2 satisfies $(1 - \partial_x)^2 v = 0$ distributionally and so has the form $(c_0 + c_1 x)e^x$, which belongs to L^2 only when it is zero. Boundedness of multiplication by D now proves the first equality in (25).

For that boundedness, the elementary continuation formula

$$\zeta(w) = \frac{w}{w-1} - w \int_1^\infty \{u\}\, u^{-w-1}\, du, \qquad \Re w > 0,$$

gives $|(w-1)\zeta(w)| \le C(1 + |w|)^2$ when $\Re w \ge 1/2$. Since $|s + 1|$ is comparable to $1 + |s|$ on the closed right half-plane, (24) is bounded and analytic there after removing its apparent pole.

Apply canonical inner–outer factorization for bounded analytic functions on the half-plane.[1] Its Blaschke zeros are precisely the stated zeros of ζ. The function D extends analytically across every finite point of the imaginary axis. A singular measure in the inner factor can therefore be supported only at its isolated boundary zeros. Such a measure would be atomic there; an atom forces exponential decay $\exp(-c/\sigma)$ as $s = \sigma + it_0$ approaches that boundary point, which is incompatible with a nonzero analytic function having a zero of finite order. Thus the finite-boundary singular measure is zero. The remaining possible inner factor is e^{-as}, $a \ge 0$. It also cannot occur with $a > 0$, since $D(x) \sim x^{-1}$ as $x \to +\infty$, whereas the other bounded factors would give $|D(x)| \le Ce^{-ax}$. Hence the inner factor is exactly B_ζ. The outer factor has dense multiplier range in H^2, so the second equality follows. □

[1] Accessible references are R. K. Srivastava, *MA650: Advanced Hardy Spaces*, IIT Guwahati lecture notes, Theorems 5.33, 5.34, 8.8 and 8.12, available from the author's university page in the official course PDF; and W. Y. Lee, *Some Recent Developments in Toeplitz Operator Theory* (2004), Theorem 1.5.4 and its proof, for the cyclic characterization of outer functions, available as an official university PDF. Rotation of the upper half-plane gives the right-half-plane convention used here.

This is a concrete version of the classical closure mechanism behind Nyman–Beurling criteria; see A. Beurling, *A closure problem related to the Riemann zeta-function*, Proc. Natl. Acad. Sci. USA 41 (1955), 312–314, https://doi.org/10.1073/pnas.41.5.312. The theorem above proves the identification for the particular theta source used here instead of assuming that it follows from a general analogy with a Hardy space.

2.8 A single omitted zero is a visible boundary mode

There is also a direct consequence valid for every fixed polynomial weight, requiring neither the Hardy factorization nor a density identification of the whole level. Let $w_s(x) = (1+x^2)^{s/2}$, $L = -\log\Omega$, and $\mathcal{H}_{s,L} = L^2([L,\infty), w_s(x)\,dx)$. Let $\mathcal{W}_{\Omega,s}$ be the closure of the core (23) in this weighted space. If ρ is a zero with $\Re\rho > 1/2$, put $z = \rho - 1/2$ and $\sigma = \Re z > 0$. Then the functional

$$\ell_{z,L}(F) = \int_L^\infty F(x)e^{-zx}\,dx \tag{29}$$

is continuous on $\mathcal{H}_{s,L}$ and vanishes on $\mathcal{W}_{\Omega,s}$. The same is true of all derivatives in z up to one less than the multiplicity of ρ. Indeed, the source functions satisfy (28) after translating the lower endpoint; analytic continuation to $\Re z > 0$ and continuity of these Laplace functionals pass the zero conditions to the closure.

The Riesz vector and its norm are explicit:

$$v_{z,L}(x) = 1_{[L,\infty)}(x)\,\frac{e^{-\overline{z}x}}{w_s(x)}, \qquad N_{z,L} = \|v_{z,L}\|^2_{\mathcal{H}_{s,L}} = \int_L^\infty \frac{e^{-2\sigma x}}{w_s(x)}\,dx.$$

As $L \to -\infty$,

$$N_{z,L} \sim \frac{e^{-2\sigma L}}{2\sigma\, w_s(L)}. \tag{30}$$

Thus its norm is not uniform in the PW level. The normalized vector escapes toward the lower endpoint and tends weakly to zero in the ambient weighted space.

Let $E_{z,L}$ be its rank-one orthogonal projector. For a test $b \in C_c^\infty((0,\infty))$, convolution preserves the supported space and

$$E_{z,L}A_b = \left(\int b(t)e^{-zt}\,dt\right) E_{z,L}.$$

Consequently its spatially cut trace is exactly

$$\operatorname{Tr}(M_{[c,\infty)}E_{z,L}A_b) = \left(\int b(t)e^{-zt}\,dt\right)\frac{\displaystyle\int_{\max\{L,c\}}^\infty e^{-2\sigma x}w_s(x)^{-1}\,dx}{\displaystyle\int_L^\infty e^{-2\sigma x}w_s(x)^{-1}\,dx}. \tag{31}$$

This is the trace of a genuine one-dimensional quotient factor, not a formal spectral sum. No additivity over nonorthogonal zero modes is being assumed.

Taking $c = \log\epsilon$ and $L = -\log\Omega$ shows the same scale as in Theorem 2.2. Assume henceforth $\Omega \to \infty$, so that $L \to -\infty$. If $\epsilon\Omega \to \tau \in (0,\infty)$, the ratio in (31) tends to

$$\begin{cases} 1, & 0 < \tau \le 1, \\ \tau^{-2\sigma}, & 1 < \tau < \infty. \end{cases}$$

It tends to 1 if $\epsilon\Omega \to 0$, and to 0 if $\epsilon\Omega \to \infty$. These limits follow by writing $x = L + y$ in the integrals and using the polynomial weight; if the displacement $c - L$ tends to $+\infty$, exponential decay controls it even when the displacement is unbounded.

Hence the cutoff coupling that preserves the local fixed-point trace also preserves this potential omitted-zero mode. A fixed lower spatial cutoff followed by PW exhaustion instead loses both. The moment condition $\int b = 0$ does not kill its eigenvalue in general. Even for b as in (15), that eigenvalue is

$$z(z^2 - 1/4) \int a(t) e^{-zt}\, dt,$$

which is nonzero for a sufficiently narrowly supported nonnegative bump a, since a nontrivial zero gives $z \notin \{0, \pm 1/2\}$. This describes exactly which support information a proposed global to local trace comparison must retain. It does not assume that such a zero exists.

2.9 A concrete approximation estimate that would complete this route

The Hardy description turns the remaining support-sensitive assertion into an explicit approximation problem on the original source functions. Normalize the Hardy norm by $\|\mathcal{L}h\|_{H^2} = \|h\|_{L^2(0,\infty)}$ and set

$$d = \inf_{F \in \mathcal{W}_1} \|e^{-x} - F(x)\|_{L^2(0,\infty)}. \tag{32}$$

Corollary 2.8 (Exact distance and a one-zero lower bound)**.** *With the notation of Theorem 2.7,*

$$d^2 = \frac{1 - |B_\zeta(1)|^2}{2}. \tag{33}$$

In particular, for every zero ρ with $\Re\rho > 1/2$, writing $z = \rho - 1/2$ gives

$$d \geq \frac{\sqrt{2\Re z}}{|1 + z|} > 0. \tag{34}$$

Equivalently, $d = 0$ if and only if there exist allowed even Schwartz sources f_j with Fourier support in $[-1, 1]$ and $f_j(0) = \widehat{f_j}(0) = 0$ such that

$$\left\| e^{-x} - e^{x/2} \sum_{n \neq 0} f_j(ne^x) \right\|_{L^2(0,\infty)} \longrightarrow 0. \tag{35}$$

This condition is equivalent to RH.

Proof. The Laplace transform of e^{-x} is the reproducing kernel $k_1(s) = (s + 1)^{-1}$, with squared norm $1/2$. Multiplication by the inner function $B = B_\zeta$ is an isometry, and its adjoint sends k_1 to $\overline{B(1)} k_1$. Hence the orthogonal projection onto BH^2 is

$$P_{BH^2} k_1 = \overline{B(1)} B k_1.$$

Its squared norm is $|B(1)|^2/2$, proving (33). Alternatively, a zero z gives the continuous evaluation condition $\mathcal{L}F(z) = 0$ on $\mathcal{W}_1$, with evaluation norm $(2\Re z)^{-1/2}$. Applied to $e^{-x} - F$, Cauchy–Schwarz gives

$$\frac{1}{|1 + z|} \leq \frac{\|e^{-x} - F\|_2}{\sqrt{2\Re z}},$$

which proves (34). The source core is dense in $\mathcal{W}_1$ by definition, so $d = 0$ is precisely (35). Finally, an inner function can have $|B(1)| = 1$ at an interior point only if it is constant. Theorem 2.7 then gives the last equivalence. □

There is a second explicit form involving only compact source functions $k \in C_c^\infty(0,\infty)$ from the proof above:

$$d^2 = \inf_{k \in C_c^\infty(0,\infty)} \frac{1}{2\pi} \int_{\mathbb{R}} \left| (i\xi - \tfrac{1}{2})\zeta(\tfrac{1}{2} + i\xi)\widehat{k}(\xi) - \frac{1}{1+i\xi} \right|^2 d\xi, \tag{36}$$

where $\widehat{k}(\xi) = \int_0^\infty k(x)e^{-i\xi x}\,dx$ in this formula. The integral is finite for every such k. This equality is Plancherel's theorem applied to the dense compact-source construction in Theorem 2.7; it does not assert that the infimum is zero.

2.10 What has been proved, and the remaining comparison

The local real-place calculation is now complete for tests supported away from norm one. Its exact boundary distribution is $H_\tau - J_\tau$ on the zero-value Paley–Wiener space, and the explicit coupling $\epsilon\Omega \to 0$ removes that distribution. The trace condition in Chapter 13 does not remove it automatically. The half-twist preserves physical support but does not remove the factor M_N in the arithmetic support bound or the escaping mass. In addition, Theorem 2.7 identifies the actual unweighted global PW closure: the extra inner factor is the Blaschke product of the zeros to the right of the critical line. Eliminating that factor is equivalent to RH, rather than an automatic consequence of support, density, or the half-twist.

To complete the rational route through the stronger support-space identification, one would need to construct the actual source approximants in (35), or prove that the explicit infimum in (36) is zero. We have proved the exact value of that infimum in terms of the inner factor and a positive lower bound from any individual omitted zero; we have not proved its vanishing. For the manuscript's weighted global trace route, an alternative comparison must account for the boundary modes in (31), rather than silently replacing the global PW projector by the local additive projector. The local trace calculation and the correct cutoff coupling are established above. Neither the arithmetic support estimates nor strong projector exhaustion provide the missing vanishing assertion.

3 Exact escaping terms, moment restrictions, and a renormalized criterion

The finite-prime degeneration can be computed explicitly. It gives a strong local convergence theorem, but the escaping part is not removed by the finite moment conditions on p. 522 of the manuscript. A natural renormalization does lead to a precise analytic criterion; its required bound is equivalent to RH, so the calculation isolates rather than proves the remaining arithmetic input.

3.1 Exact Mellin transforms and the two boundary norms

For $X \geq 2$ write

$$Q_X = \prod_{p \leq X} p, \qquad E_X(z) = \prod_{p \leq X} (1 - p^{-z}), \qquad P_X = \prod_{p \leq X} (I - T_{\log p}), \quad T_h f(x) = f(x+h).$$

The scalar Euler product E_X must be distinguished from the operator P_X. Its zeros are $2\pi ik/\log p$, $p \leq X$, $k \in \mathbb{Z}$: all nonzero zeros are simple, and zero has multiplicity $\pi(X)$. Distinct prime lattices meet only at zero. Put

$$f_\infty(u) = (3u^2 - 2\pi u^4)e^{-\pi u^2}, \quad K(x) = 2\sum_{n \geq 1} f_\infty(ne^x), \quad g(x) = 2f_\infty(e^x), \quad K_X = P_X K,$$

and denote the escaping difference by $D_X = K_X - g$. We use the Mellin convention $\mathcal{M}h(z) = \int_{\mathbb{R}} h(x)e^{zx}\,dx$.

Proposition 3.1 (The escaping Mellin transform)**.** *Let*

$$F(z) = \frac{z(1-z)}{2}\pi^{-z/2-1}\Gamma(z/2).$$

Then, for every finite X,

$$\mathcal{M}g(z) = F(z) \qquad (\Re z > -2), \tag{37}$$

$$\mathcal{M}K_X(z) = F(z)\zeta(z)E_X(z) = -\frac{\xi_\zeta(z)}{\pi}E_X(z) \qquad (z \in \mathbb{C}), \tag{38}$$

$$\mathcal{M}D_X(z) = F(z)\big(\zeta(z)E_X(z) - 1\big) \qquad (\Re z > -2). \tag{39}$$

The second line is entire. The first and third have meromorphic continuations with possible poles only at $-2, -4, \ldots$; for the third these are precisely the negatives of the poles of F. At a nontrivial zero ρ of multiplicity m_ρ,

$$(\mathcal{M}D_X)^{(j)}(\rho) = -F^{(j)}(\rho) \quad (0 \le j < m_\rho), \qquad F(\rho) \neq 0.$$

At the two real normalization points,

$$\mathcal{M}D_X(0) = -\frac{1}{\pi}, \qquad \mathcal{M}K_X(1) = -\frac{E_X(1)}{2\pi}.$$

Proof. Gaussian integration gives the first identity. The apparent pole of F at zero cancels: $F(0) = 1/\pi$. Its zero at one is simple, and its other poles are the simple gamma poles at the negative even integers. The identity $\mathcal{M}K = F\zeta = -\xi_\zeta/\pi$ follows by termwise integration for $\Re z > 1$ and then by analytic continuation. Poisson summation, with $f_\infty(0) = \int f_\infty = 0$, shows that K and its finite translates decrease faster than every exponential at both ends. The shift T_h multiplies the Mellin transform by e^{-zh}, which gives the factor E_X. At a nontrivial zero, E_X is nonzero because its zeros lie on $\Re z = 0$, and F is holomorphic and nonzero. The jet assertion follows. Finally $E_X(0) = 0$ and $\lim_{z\to 1} F(z)\zeta(z) = -1/(2\pi)$. □

For the actual half twist, put $b = 1/2$ and $U_b h(x) = e^{bx}h(x)$; the following formulas also hold for $0 < b < 1$. Let

$$H_{\alpha,r} = L^2\big(\mathbb{R}, e^{2\alpha|x|}(1+x^2)^{r/2}\,dx\big), \qquad 0 < \alpha < b, \quad r \ge 0.$$

The Mellin functional

$$\ell_z(v) = \int_{\mathbb{R}} v(x)e^{(z-b)x}\,dx$$

is continuous on this space whenever $|\Re z - b| < \alpha$, by Cauchy–Schwarz. Its jth derivative in z remains continuous at a boundary $|\Re z - b| = \alpha$ exactly when $r > 2j + 1$.

There is an exact boundary-line description, up to the unavoidable equivalence between the maximum and the sum of the two weights. For $r = 0$, let $N_X = \|U_b D_X\|_{H_{\alpha,0}}$ and put

$$B_X = \frac{1}{2\pi}\sum_{\sigma=b-\alpha,b+\alpha}\int_{\mathbb{R}} |F(\sigma+it)|^2|\zeta(\sigma+it)E_X(\sigma+it) - 1|^2\,dt.$$

Plancherel gives

$$N_X^2 \le B_X \le 2N_X^2. \tag{40}$$

For $r = 2m$, the same assertion holds with each integrand replaced by

$$\sum_{j=0}^{m} \binom{m}{j} \left| \partial_t^j \left[F(\sigma+it)(\zeta(\sigma+it)E_X(\sigma+it) - 1) \right] \right|^2 .$$

Thus passing from local convergence to this Hilbert norm requires control of actual boundary values, not only pointwise convergence on a spatial half-line. In particular, if a nontrivial zero ρ lies in $|\Re\rho - b| < \alpha$, then

$$\|U_b D_X\|_{H_{\alpha,r}} \geq |F(\rho)|/\|\ell_\rho\| > 0$$

for every X. This statement is conditional only on the indicated zero belonging to the strip; it makes no new zero-location assumption.

3.2 Fast local convergence and global divergence after every nonzero test

Theorem 3.2 (The escape persists under finite moment filters)**.** *For each fixed $L \in \mathbb{R}$ and the parameters above there are $C_L, c_L > 0$ such that*

$$\|\mathbf{1}_{[L,\infty)} U_b D_X\|_{H_{\alpha,r}} \leq C_L e^{-c_L X^2}.$$

For any nonzero $a \in C_c^\infty(\mathbb{R})$, define the bounded convolution operator

$$A_a v(x) = \int_{\mathbb{R}} a(u) v(x-u)\, du, \qquad \mathcal{L}a(w) = \int_{\mathbb{R}} a(u) e^{wu}\, du.$$

Then

$$\|A_a U_b K_X\|_{H_{\alpha,r}} \longrightarrow \infty, \qquad \|A_a U_b D_X\|_{H_{\alpha,r}} \longrightarrow \infty.$$

These conclusions hold even if a has any prescribed finite number of vanishing moments $\int u^j a(u)\, du = 0$. For every nontrivial zero in the open strip one also has the exact filtered obstruction

$$\ell_\rho(A_a U_b D_X) = -\mathcal{L}a(\rho - b) F(\rho).$$

Proof. Finite inclusion–exclusion gives

$$K_X(x) = 2 \sum_{\substack{n \geq 1 \\ (n, Q_X)=1}} f_\infty(ne^x).$$

Every surviving $n \geq 2$ exceeds X. Since $|f_\infty(y)| \leq Ce^{-\pi y^2/2}$, for $x \geq L$,

$$|D_X(x)| \leq C_L \exp(-\pi X^2 e^{2x}/4).$$

Splitting the squared exponential into two equal factors bounds the weighted squared integral by $C_L \exp(-\pi X^2 e^{2L}/4)$. One may therefore take $c_L = \pi e^{2L}/8$. Compact convolution preserves convergence on every such half-line, by enlarging L by the radius of the support of a.

Compact convolution is bounded on $H_{\alpha,r}$ because a translation by u has norm at most $C_r e^{\alpha|u|}(1+u^2)^{r/4}$. Choose a real number

$$c \in (b-\alpha, b+\alpha) \cap (0,1) \quad \text{with} \quad \mathcal{L}a(c-b) \neq 0.$$

This is possible because the Laplace transform of a nonzero compactly supported smooth function is not identically zero and has isolated zeros. The elementary Euler-product estimate

$$0 < E_X(c) \leq E_X(1) \leq \left(\sum_{n \leq X} \frac{1}{n} \right)^{-1} \longrightarrow 0$$

uses only that every integer $n \le X$ has all its prime factors at most X. Suppose that $A_a U_b K_X$ had a bounded subsequence in the Hilbert space. A weakly convergent further subsequence would have limit $A_a U_b g$, because local strong convergence fixes the limit on every compact interval. But continuity of ℓ_c and Proposition 3.1 would give

$$\ell_c(A_a U_b g) = \lim_X \mathcal{L}a(c-b)F(c)\zeta(c)E_X(c) = 0,$$

whereas its actual value is $\mathcal{L}a(c-b)F(c) \neq 0$. This is a contradiction. The absence of any bounded subsequence proves divergence to infinity; subtracting the fixed vector $A_a U_b g$ proves the assertion for D_X. Finally, compact convolution multiplies ℓ_z by $\mathcal{L}a(z-b)$, proving the displayed filtered-zero formula. □

The same proof with no convolution, taking $c = b$, shows $\|U_b K_X\|_{H_{\alpha,r}}, \|U_b D_X\|_{H_{\alpha,r}} \to \infty$.

This argument uses neither PNT nor RH. It shows that finitely many moment conditions cannot supply the missing global vector bound, even though they do remove the particular constant-mode discrepancy in the finite orbital trace calculation. The conclusion concerns these vectors and convolutions; it is not a claim that every possible renormalized trace construction must use such a bound.

Proposition 3.3 (An explicit profile at the first escaping scale)**.** *Let $0 < b < 1$, $0 < \alpha < b$, and $r \ge 0$, as above. Unconditionally, using PNT, locally uniformly for $y \in \mathbb{R}$,*

$$\frac{\log X}{X^{1-b}} U_b K_X(y - \log X) \longrightarrow h_b(y) := -2e^{(b+2)y} e^{-\pi e^{2y}}.$$

The same limit holds with K_X replaced by D_X. After applying A_a, the limit is $a * h_b$. If I is any bounded interval, then

$$\frac{(\log X)^{2-r}}{X^{2(1-b+\alpha)}} \|\mathbf{1}_{I-\log X} A_a U_b D_X\|^2_{H_{\alpha,r}} \longrightarrow \int_I |(a * h_b)(y)|^2 e^{-2\alpha y}\, dy.$$

For every nonzero $a \in C_c^\infty(\mathbb{R})$ there is consequently $c_a > 0$ such that, for all sufficiently large X,

$$\|A_a U_b K_X\|_{H_{\alpha,r}},\ \|A_a U_b D_X\|_{H_{\alpha,r}} \ge c_a X^{1-b+\alpha} (\log X)^{r/2-1}.$$

Proof. Write $v = e^y$ in a fixed compact subinterval of $(0, \infty)$. Every composite integer coprime to Q_X is greater than X^2. Its total contribution to $K_X(y - \log X)$ is bounded by $C \sum_{n > X^2} e^{-cn^2/X^2} = O(e^{-c'X^2})$, uniformly on that compact set. The term $n = 1$ is $O(X^{-2})$. The remaining terms are exactly $2\sum_{p>X} f_\infty(vp/X)$. PNT gives

$$\frac{\log X}{X} \sum_{p>X} f_\infty(vp/X) \longrightarrow \int_1^\infty f_\infty(vt)\, dt = -v^2 e^{-\pi v^2}.$$

For completeness, on a bounded t interval this is weak convergence of the rescaled prime-counting measures, uniformly for the compact family of functions $f_\infty(vt)$. The Gaussian tails are uniform: PNT's elementary upper consequence $\pi(t) \le Ct/\log t$ bounds the rescaled mass in $[j, j+1]$ by $C(j+1)$, and the integrands there are $O(e^{-cj^2})$. The last integral is explicit because $f_\infty(u) = (u^3 e^{-\pi u^2})'$. Multiplication by the half twist gives the claimed profile, and compact convolution preserves the local uniform convergence.

On the moving interval $x = y - \log X$ the weight satisfies, uniformly for $y \in I$,

$$e^{2\alpha|x|}(1 + x^2)^{r/2} = (1 + o(1))X^{2\alpha}(\log X)^r e^{-2\alpha y}.$$

This proves the norm limit. Finally

$$\widehat{h}_b(t) = -\pi^{-1-(b-it)/2}\Gamma(1+(b-it)/2)$$

has no real zeros. Hence $a * h_b$ is not identically zero if $a \neq 0$; choose a bounded interval where its squared integral is positive. The lower bound follows, and the identical moving profile for K_X gives the other lower bound. □

This profile describes a specific window moving toward $-\infty$, not the whole escaping function. Contributions on still more distant scales may be larger and cancel Mellin moments of this first window.

One can also quantify substantial subsequential growth by PNT. For fixed $z = b + it$, $0 < b < 1$, the analytic logarithm on $\Re z > 0$ satisfies

$$\log E_X(z) = -\frac{X^{1-z}}{(1-z)\log X} + o_z\left(\frac{X^{1-b}}{\log X}\right). \tag{41}$$

Indeed PNT and partial summation give this asymptotic for $-\sum_{p\le X} p^{-z}$, while the terms with prime-power exponent at least two are $O_b(1+\log X + X^{\max(1-2b,0)})$, a smaller order. Choose small $t > 0$ with $\zeta(b+it) \neq 0$. Such a choice exists since $\zeta(b) \neq 0$ for $0 < b < 1$, as follows from the alternating Dirichlet series. Along real cutoffs for which $\arg(1/(1-z)) - t\log X$ is an odd multiple of π,

$$\log|E_X(z)| = (1+o(1))\frac{X^{1-b}}{|1-z|\log X}.$$

The continuous functional ℓ_z therefore gives the same expression as a lower bound for $\log\|U_b K_X\|_{H_{\alpha,r}}$, up to lower-order terms. At $b = 1/2$ this is exponential growth on the scale $\sqrt{X}/\log X$. The use of PNT here is unconditional; for the classical PNT and its relation to $\psi(x)$ see DLMF, Section 25.16(i).

3.3 Which zero poles survive a moment-restricted trace?

The raw transform in (39) has nonzero *values* at nontrivial zeros, not poles. Its divisor has a separate exact description:

$$L_X(z) := \frac{d}{dz}\log\frac{\mathcal{M}K_X(z)}{F(z)} = \frac{\zeta'}{\zeta}(z) + \sum_{p\le X}\frac{\log p}{p^z-1}. \tag{42}$$

Every nontrivial zero ρ is a pole of L_X of residue m_ρ, for every X. The pole at one has residue -1. The added Euler terms have poles only on $\Re z = 0$: residue one at each nonzero finite-prime lattice point, and residue $\pi(X)$ at zero. The remaining poles are the trivial zeta zeros $-2, -4, \ldots$, each with residue one. This lists all the poles, since zeta has no zeros on $\Re z = 0$. In $\Re z > 1$ one instead has the absolutely convergent tail representation

$$L_X(z) = -\sum_{p>X}\sum_{k\ge1}(\log p)p^{-kz}.$$

Its inverse Laplace distribution is supported on $u = k\log p > \log X$. Pairing this distribution with any fixed compact test on $u > 0$ is consequently zero once X exceeds the exponential of the support endpoint. Nevertheless, for a small circle about any fixed nontrivial zero ρ,

$$\frac{1}{2\pi i}\oint L_X(z)\,dz = m_\rho$$

for every X. Thus the local test limit, the limit in $\Re z > 1$, and meromorphic continuation across the zero set cannot be interchanged. Multiplying the finite Euler product by an exponential of an entire function adds an entire logarithmic derivative and does not alter these residues.

Here is a precise version of the corresponding spectral assertion. Let $\lambda_\rho = \rho - 1/2$ and let $0 \le r_\rho \le m_\rho$. On positive tests set

$$R_r(a) = \sum_\rho r_\rho \mathcal{L}a(\lambda_\rho), \qquad a \in C_c^\infty((0,\infty)).$$

The usual zero-counting bound and integration by parts ensure absolute convergence. For a polynomial $q(z) = \prod_\nu (z-\nu)^{d_\nu}$, let

$$\mathcal{A}_q = \left\{ a : (\mathcal{L}a)^{(j)}(\nu) = 0 \text{ for every } \nu \text{ and } 0 \le j < d_\nu \right\}.$$

Proposition 3.4 (Finite moments hide only their prescribed modes)**.** *The distribution R_r vanishes on $\mathcal{A}_q$ if and only if $r_\rho = 0$ for every ρ with $q(\lambda_\rho) \neq 0$. In particular, for $q(z) = z^m$, every noncentral omitted zero remains detectable. If r_ρ is supported on $|\Re\lambda_\rho| \ge \alpha > 0$, then vanishing on this finite-moment test space is equivalent to the complete absence of the omitted zeros.*

Proof. For every $v \in C_c^\infty((0,\infty))$, $a = q(-\partial_u)v$ belongs to $\mathcal{A}_q$ and

$$R_r(a) = \sum_\rho r_\rho q(\lambda_\rho)\mathcal{L}v(\lambda_\rho).$$

The meromorphic detection argument applies to these coefficients, even though they need not be nonnegative: their growth is at most polynomial in the zero height. Explicitly, choose a smooth η equal to zero before $b_0 > 0$ and one after $b_1 > b_0$, and put

$$J(w) = \int \eta'(u) e^{-wu}\, du, \qquad H(w) = J(w)/w.$$

The series

$$\sum_\rho r_\rho q(\lambda_\rho) H(z - \lambda_\rho)$$

is locally normally convergent away from its discrete poles: J has arbitrarily rapid vertical decay, which absorbs both the polynomial q and the zero-counting bound. Its residue at λ_ρ is $r_\rho q(\lambda_\rho)$. Testing with compact truncations of $\eta(u)e^{-zu}$ and then sending the cutoff to infinity in $\Re z > 1/2$ shows that the series is zero there whenever R_r vanishes on $\mathcal{A}_q$. Meromorphic continuation forces every residue to vanish. Conversely, if only roots of q can carry nonzero r_ρ, all its terms vanish on $\mathcal{A}_q$ by definition. □

For the trivial rational isotype, manuscript (13.3.1.2), p. 522, is exactly $\int_0^\infty a(u)\, du = 0$ after $u = \log t$ and restriction to positive support. Thus $q(z) = z$. In a fixed normalized isotype over a degree-d field, after canceling its phase along the chosen norm section, the displayed derivative conditions give $q(z) = z^d$. They can remove a contribution supported at the central mode $\lambda = 0$; they cannot remove an off-critical zero. This rescues the constant-mode cancellation in the local shell sum, but it does not make the required global trace identity a consequence of that cancellation.

3.4 A natural renormalization and its exact arithmetic strength

The main prime density suggests removing the leading divergence from the Euler product. Define the entire functions

$$I_X(z) = \int_2^X \frac{t^{-z}}{\log t}\, dt, \qquad C_X(z) = e^{I_X(z)} E_X(z),$$

and the entire, nowhere-zero function

$$B(z) = \frac{e^{-\gamma_E}}{\log 2} \exp\left[-\sum_{k\geq 1} \frac{(-1)^k((z-1)\log 2)^k}{k\,k!} \right].$$

Here γ_E is Euler's constant. This renormalization is an explicit candidate, not an identification with any particular operator in the manuscript.

Theorem 3.5 (A locally bounded renormalization is equivalent to RH)**.** *The following are equivalent:*

1. *The Riemann hypothesis for ζ.*
2. *The family $\{C_X : X \geq 2\}$ is locally bounded on $\Re z > 1/2$.*
3. *The functions C_X converge locally uniformly on $\Re z > 1/2$.*

When they hold, the limit is

$$C(z) = \frac{B(z)}{(z-1)\zeta(z)}.$$

The singularity at one is removable, with $C(1) = e^{-\gamma_E}/\log 2$. Using the classical consequence $\psi(x) = x + O(\sqrt{x}\log^2 x)$ of RH, for each compact $\mathcal{K}$ in this half-plane, with $\sigma_{\mathcal{K}} = \min_{z\in\mathcal{K}} \Re z$,

$$\sup_{z\in\mathcal{K}} |C_X(z) - C(z)| = O_{\mathcal{K}}\Big(X^{1/2-\sigma_{\mathcal{K}}} \log X \Big).$$

The same rate holds for the canonically chosen analytic logarithms.

Proof. For $\Re z > 1$ the Euler product gives $C_X(z) \to e^{I_\infty(z)}/\zeta(z)$, locally uniformly. Substitution $u = \log t$ and the elementary expansion of the exponential integral give

$$(z-1)e^{I_\infty(z)} = B(z).$$

If the family is locally bounded on $\Re z > 1/2$, Montel's theorem and uniqueness of the limit on the smaller half-plane give a holomorphic limit C throughout $\Re z > 1/2$. The identity theorem yields

$$(z-1)\zeta(z)C(z) = B(z).$$

Because B has no zeros, ζ has no zeros in this half-plane. The functional equation then gives RH. This proves (2) $\Rightarrow$ (1) and also shows that the limit, if it exists, is forced.

For the converse, use the standard RH consequence $\psi(t) - t = O(t^{1/2}\log^2 t)$. This implication follows, for example, from the truncated explicit formula; a primary reference with a stronger explicit estimate is Lee–Nosal, *Sharper bounds for the error in the prime number theorem assuming the Riemann Hypothesis*. Put

$$L_X^*(z) = \sum_{p^k \leq X} \frac{p^{-kz}}{k} = \int_{2^-}^{X} \frac{t^{-z}}{\log t}\, d\psi(t).$$

The difference between this prime-power truncation and the logarithm of the finite-prime product is

$$\log E_X(z) = -L_X^*(z) - T_X(z), \qquad T_X(z) = \sum_{\substack{p\leq X,\ k\geq 2 \\ p^k > X}} \frac{p^{-kz}}{k}.$$

For $\Re z \geq \sigma_0 > 1/2$,

$$|T_X(z)| \leq C_{\sigma_0} X^{1/2-\sigma_0}. \tag{43}$$

To see this, primes $p \leq \sqrt{X}$ contribute at most $C_{\sigma_0} X^{-\sigma_0}$ each, since their first retained power is greater than X; there are at most $\sqrt{X}$ of them. For larger primes, the sum of all powers $k \geq 2$ is bounded by $C_{\sigma_0} p^{-2\sigma_0}$, and comparison with the integer integral gives the same bound.

Let $\mathcal{E}(t) = \psi(t) - t$ and $f_z(t) = t^{-z}/\log t$. Then

$$I_X(z) - L_X^*(z) = -\int_{2^-}^{X} f_z(t)\, d\mathcal{E}(t).$$

Stieltjes integration by parts gives a locally uniformly convergent limit for $\Re z > 1/2$: the upper boundary is $O_{\mathcal{K}}(X^{1/2-\sigma_{\mathcal{K}}} \log X)$, and the tail integral of $f_z'(t)\mathcal{E}(t)$ has the same bound. Equation (43) proves convergence of $\log C_X = I_X - L_X^* - T_X$ with the stated rate. Exponentiating gives (1) $\Rightarrow$ (3) and its quantitative assertion. Local uniform convergence implies local boundedness, completing the equivalence. □

The standard error estimate is the arithmetic input in this proof, not an estimate supplied by the theta or Hilbert-space construction. More generally, an input $\psi(x) - x = O(x^\theta \log^A x)$ gives the same argument only in $\Re z > \max\{\theta, 1/2\}$. Ordinary PNT therefore does not cross the line $\Re z = 1$ in this argument.

Even under RH, this renormalization cannot be locally bounded across a nontrivial zero on the critical line: its forced continuation has a pole there, since B is nowhere zero. To cross that boundary one must retain the spectral residues explicitly. The theorem establishes an exact criterion for a natural attempted repair; it does not establish its RH-equivalent bound.

3.5 Why ordinary Möbius cancellation does not bound these finite products

Expanding the finite Euler product gives a divisor-box truncation,

$$E_X(z) = \sum_{d \mid Q_X} \mu(d) d^{-z}, \qquad M_X(y) = \sum_{\substack{d \leq y \\ d \mid Q_X}} \mu(d).$$

It is not the ordinary partial sum over $d \leq X$. Since $M_X(y) = 0$ for $y \geq Q_X$, Abel summation gives the entire identity

$$E_X(z) = z \int_1^\infty M_X(y) y^{-z-1}\, dy.$$

Proposition 3.6 (No uniform sublinear Möbius bound for the prime box)**.** *For no $\theta < 1$ is there a constant C such that $|M_X(y)| \leq C y^\theta$ for every $X \geq 2$ and $y \geq 1$. In particular a uniform square-root bound for these finite-prime coefficients is false, whether or not RH holds.*

Proof. Such a bound would make the displayed integrals locally bounded and convergent on $\Re z > \theta$, because $M_X(y) \to M(y)$ for each fixed y. On $\Re z > 1$ the limit equals $1/\zeta(z)$. Hence its analytic continuation H satisfies $\zeta(z)H(z) = 1$ wherever ζ is holomorphic in this connected half-plane. Choose real $\max\{\theta, 0\} < c < 1$. But $E_X(c) \to 0$ by the elementary estimate used in Theorem 3.2, so $H(c) = 0$, contradicting $\zeta(c)H(c) = 1$. □

For example, when $X < y \leq 2X$, the omitted integers up to y are exactly the primes in $(X, y]$, and $M_X(y) = M(y) + \pi(y) - \pi(X)$. Thus importing a bound for the ordinary Möbius sum into the prime-box

sum omits a real arithmetic term. The surviving constructive results are the exact local convergence, the Mellin and divisor formulas, and the moment-restricted detector. They specify which escaping information a valid global trace bridge must retain; none permits discarding that information by finite moments alone.

4 Testing a positivity completion: a negative witness and the missing estimate

The remaining identity cannot be obtained merely by calling a kernel an orthogonal compression. We test that possibility directly below. The result is a concrete alternative target: a support-dependent lower bound for the full Weil quadratic form. We prove the estimate supplied by elementary unconditional bounds, and construct a witness showing exactly how much stronger an estimate would have to be to prove GRH.

4.1 Finite moment conditions do not conceal off-strip zeros

Let ψ be a fixed normalized unitary Hecke character, let ρ run over the nontrivial zeros of its completed L-function, and put $\lambda_\rho = \rho - \frac{1}{2}$, with multiplicity m_ρ. We use the usual strip bound $|\Re\lambda_\rho| \le \frac{1}{2}$, the polynomial zero-counting bound, and the reflection symmetry $\lambda \mapsto -\overline{\lambda}$ for this zero multiset. For a multiplicative test set $\mathcal{M}f(z) = \int_0^\infty f(t)t^z\,dt/t$. Let $B \subset \mathbb{C}$ be finite and let $d_\beta \ge 1$ be integers. On positive-half-line tests define

$$\mathcal{A}_{B,d} = \{f \in C_c^\infty((1,\infty)) : (\mathcal{M}f)^{(j)}(\beta) = 0,\ \beta \in B,\ 0 \le j < d_\beta\}.$$

Proposition 4.1 (The constrained detector)**.** *Suppose $0 \le r_\rho \le m_\rho$. The functional $R_r(f) = \sum_\rho r_\rho \mathcal{M}f(\lambda_\rho)$ vanishes on $\mathcal{A}_{B,d}$ if and only if $r_\rho = 0$ at every $\lambda_\rho \notin B$.*

Proof. The stated Mellin-jet functionals are linearly independent: in the coordinate $u = \log t$ this is the linear independence on an open interval of the exponential polynomials $u^j e^{\beta u}$. A linear functional vanishing on their joint kernel is consequently a finite linear combination of those functionals. If R_r vanishes there, write

$$R_r(f) = \sum_{\beta\in B}\sum_{j<d_\beta} c_{\beta,j}(\mathcal{M}f)^{(j)}(\beta).$$

Apply the fixed-cutoff argument of Proposition 3.4 to both sides; its proof uses only the strip and zero-counting bounds and therefore applies to this fixed Hecke character as well. Initially take $\operatorname{Re} z > \frac{1}{2}$ and $\operatorname{Re} z > \operatorname{Re}\beta$ for every $\beta \in B$. It gives a meromorphic identity

$$\sum_\rho r_\rho H(z-\lambda_\rho) = \sum_{\beta,j} c_{\beta,j}(-1)^j H^{(j)}(z-\beta).$$

The left side has residue r_ρ at λ_ρ. The right side has poles only in B. This proves the required vanishing outside B. In fact its higher-order pole coefficients must also vanish, since the left side has only simple poles. Conversely, if the remaining spectral points belong to B, the zeroth-order moment conditions kill every remaining term. □

In a fixed character isotype, the conditions (13.3.1.2) on manuscript p. 522 become $(\mathcal{M}f)^{(j)}(0) = 0$ for $0 \le j < [F:\mathbb{Q}]$. They therefore do not conceal any zero outside a positive-width strip. For the formula on the critical line in Theorem 13.3.1.1, the same argument detects both off-line zeros and any excess

multiplicity at a critical zero other than $\lambda = 0$. In particular, the stated cap $\min(m_\rho, [F : \mathbb{Q}])$ is an additional multiplicity assertion away from that constrained point. For $F = \mathbb{Q}$ it asserts simplicity there, which is stronger than RH. In the zeta case the exceptional point is not a zero: $\zeta(\frac{1}{2}) = \eta(\frac{1}{2})/(1-\sqrt{2}) < 0$, since the alternating Dirichlet series $\eta(\frac{1}{2})$ is positive. Thus its cap of one would assert simplicity of every nontrivial zeta zero as well as RH. Removing a constant zero-frequency shell term by a moment condition does not remove this off-strip obstruction.

4.2 An explicit compactly supported witness for failure of Weil positivity

Work in the logarithmic variable $u \in \mathbb{R}$. For $g \in C_c^\infty(\mathbb{R})$ define

$$G_g(z) = \int_{\mathbb{R}} g(u) e^{zu}\, du, \qquad g^*(u) = \overline{g(-u)}, \qquad \mathcal{B}_\psi(g) = \sum_\rho m_\rho G_g(\lambda_\rho)\overline{G_g(-\overline{\lambda_\rho})}.$$

This is the full zero distribution evaluated on $g * g^*$. The sum converges absolutely. The functional equation and complex conjugation give invariance of the multiset under $\lambda \mapsto -\overline{\lambda}$, with equal multiplicities, so $\mathcal{B}_\psi(g)$ is real. Under GRH it is the nonnegative sum $\sum_\rho m_\rho |G_g(\lambda_\rho)|^2$.

Theorem 4.2 (A Gaussian witness, with exact finite constraints)**.** *Suppose the completed zero set contains $\lambda_+ = a + i\gamma$ with $a > 0$. Its reflected zero is $\lambda_- = -a + i\gamma$, with the same multiplicity m. Let B be any finite set not containing λ_+ or λ_-, and prescribe any finite orders of vanishing at its points. There are $g_T \in C_c^\infty(\mathbb{R})$, $T \geq 1$, satisfying all these Mellin-jet conditions exactly, such that*

$$\operatorname{supp} g_T \subset [-2T - 1, 2T + 1], \tag{44}$$

$$\mathcal{B}_\psi(g_T) \longrightarrow -2m, \tag{45}$$

$$\|g_T\|^2_{W^{N,2}(\mathbb{R})} \leq C_N T^{-1/2} e^{-2a^2 T} \quad \textit{for every fixed integer } N \geq 0. \tag{46}$$

Here $\|g\|^2_{W^{N,2}} = \sum_{j=0}^N \|g^{(j)}\|_2^2$; this is an ordinary Sobolev norm, not the polynomially weighted norm denoted H_s earlier.

Proof. There are only finitely many distinct zeros with $|\operatorname{Im}\lambda - \gamma| < 1$. Polynomial interpolation gives a polynomial p such that

$$p(\lambda_+) = 1, \qquad p(\lambda_-) = -1,$$

which vanishes at all other zeros in that band and to the prescribed orders at B. Coincident vanishing conditions are combined by taking the larger required order. The excluded target points make all the interpolation conditions consistent.

First consider the entire function

$$F_T(z) = p(z) \exp\big(T((z - i\gamma)^2 - a^2)\big).$$

It equals $1, -1$ at the target pair and zero at every other zero in the chosen band. Writing $z = \sigma + i\tau$, its squared modulus is

$$|F_T(z)|^2 = |p(z)|^2 e^{2T(\sigma^2 - (\tau-\gamma)^2 - a^2)}.$$

At the remaining zeros, $|\sigma| \leq \frac{1}{2}$ and $|\tau - \gamma| \geq 1$. Polynomial growth of p, Gaussian decay, and the unconditional zero-counting bound show that

$$\sum_{\rho \notin \{\rho_+, \rho_-\}} m_\rho |F_T(\lambda_\rho)|^2 = O(e^{-T}).$$

Consequently the quadratic spectral sum for F_T is $-2m + O(e^{-T})$, by Cauchy–Schwarz and reflection invariance. The imaginary oscillation has thus been removed without asserting that individual off-line terms have a definite sign.

The inverse bilateral Laplace transform is explicit. Put

$$q_T(u) = (4\pi T)^{-1/2} e^{-u^2/(4T)} e^{-i\gamma u}.$$

Its transform is $e^{T(z-i\gamma)^2}$, and multiplication of the transform by $p(z)$ corresponds to $p(-\partial_u)$. Choose $\chi_T \in C_c^\infty(\mathbb{R})$ which is one on $[-2T, 2T]$, zero outside $[-2T-1, 2T+1]$, and has derivative bounds independent of T. Define

$$g_T(u) = e^{-a^2 T} p(-\partial_u)(\chi_T(u) q_T(u)).$$

Putting the cutoff *inside* this differential operator is essential: the transform retains the exact factor $p(z)$, so the prescribed finite conditions hold exactly after compactification.

Let $\widetilde{F}_T = G_{g_T}$. For $|\sigma| \le \frac{1}{2}$ and $|u| \ge 2T$,

$$-\frac{u^2}{4T} + \sigma u \le 0,$$

and the exponent decreases outwards with slope of absolute value at least $\frac{1}{2}$. Every derivative of $(\chi_T - 1) q_T e^{\sigma u}$ therefore has L^1 norm bounded by $C_k T^{-1/2}$, uniformly in $T \ge 1$ and σ in this strip. After integrating by parts sufficiently many times to absorb the degree of p, we obtain for every $L \ge 0$

$$|\widetilde{F}_T(\sigma + i\tau) - F_T(\sigma + i\tau)| \le C_L e^{-a^2 T} (1 + |\tau|)^{-L}.$$

The difference tends to zero in the weighted sequence space $\ell^2(\mathcal{Z}_\psi, m_\rho)$. The sequence of values of F_T at the zeros has bounded norm there, as already proved. The quadratic sums for F_T and $\widetilde{F}_T$ thus differ by $o(1)$, proving (45).

Finally, every fixed derivative of q_T has L^2 norm $O(T^{-1/4})$ for $T \ge 1$; derivatives of the cutoff contribute only Gaussian tails near $|u| = 2T$. The fixed polynomial differential operator and the factor $e^{-a^2 T}$ give (46). The support assertion follows directly from the construction. □

This also proves Weil's criterion on a test subspace defined by finitely many Mellin conditions, provided each off-line reflection pair avoids the prescribed constraint points. In particular, constraints at $z = 0$ and at the completed poles $z = \pm\frac{1}{2}$ do not change its implication to RH. Conditions on G_g at zero also give the required conditions on the Mellin transform of $g * g^*$, since that transform is $G_g(z)\overline{G_g(-\overline{z})}$. The pole conditions make the explicit formula's polar contribution zero. Connes–Consani's *Weil positivity and Trace formula, the archimedean place*, Appendix C, Proposition C.1, proves the corresponding finite-condition criterion for zeta. The argument above supplies an explicit witness and a support/norm estimate for the present application.

For an omitted-zero multiset consisting of the zeros with $|\Re\lambda_\rho| \ge \alpha > 0$, use the same construction with their multiplicities. If it is nonempty, its quadratic form has a negative direction; replacing the target value -1 by 1 also gives a positive direction. Thus its quadratic form is indefinite unless the omitted-zero correction is zero.

4.3 An actual lower-bound calculation, and the size of the missing improvement

Corollary 4.3 (A quantitative sufficient estimate)**.** *Suppose, on the finitely constrained test space just described, that for a fixed integer $N \ge 0$ and constants $C, c \ge 0$ one could prove*

$$\mathcal{B}_\psi(g) \ge -C e^{cR} \|g\|_{W^{N,2}}^2, \qquad \operatorname{supp} g \subset [-R, R], \quad R \ge 1. \tag{47}$$

Then no off-line zero whose reflected pair avoids B can have $|\operatorname{Re}\rho - \frac{1}{2}|^2 \geq c$. In particular, a bound independent of R, or bounds of this form for every $c > 0$ with constants allowed to depend on c, imply GRH when the constrained points contain no off-line zeros. Conversely, GRH gives these lower bounds with $C = 0$.

Proof. Insert the witness of Theorem 4.2 and take $R = 2T + 1$. The absolute value of the possible negative lower bound is at most $C'C_N T^{-1/2} e^{-2(a^2-c)T}$, which tends to zero if $a^2 \geq c$, including equality. This contradicts the limiting value $-2m$. Taking $c < a^2$ proves the subexponential version. □

Equivalently, let $E(R)$ be the infimum of $\mathcal{B}_\psi(g)$ over unit L^2-norm constrained tests supported in $[-R, R]$. An off-line zero at distance a from the critical line forces, for all large T,

$$E(2T + 1) \leq -c_a \sqrt{T}\, e^{2a^2 T}.$$

This follows by normalizing the same witness. Thus proving that these lowest Rayleigh values stay bounded below, much less converge to zero, would exclude all off-line zeros. Semiboundedness on each fixed interval does not give this assertion as the interval grows.

Here is what a direct unconditional estimate actually supplies. For zeta impose $G_g(\pm\frac{1}{2}) = 0$ and write $f = g * g^*$ in logarithmic coordinates. The correctly normalized Archimedean explicit formula, with

$$k(u) = \frac{e^{-u/2}}{1 - e^{-2u}}, \qquad C_0 = \log(8\pi) + \gamma_{\mathrm{E}} + \frac{\pi}{2},$$

can be rearranged into the exact identity

$$\mathcal{B}_1(g) = \int_0^\infty k(u)\|g - T_u g\|_2^2 \, du - C_0\|g\|_2^2 - 2\sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \operatorname{Re} f(\log n). \tag{48}$$

Here $\Lambda(n)$ is the von Mangoldt function and γ_{E} is Euler's constant. To check the constant, start with the Archimedean formula in Connes–Consani, equation (3), and use

$$2\operatorname{Re} f(u) = 2\|g\|_2^2 - \|g - T_u g\|_2^2, \qquad 2\int_0^\infty (1 - e^{-u/2})k(u)\, du = \log 2 + \frac{\pi}{2}.$$

The integral in (48) is nonnegative and finite for smooth compactly supported g. If $\operatorname{supp} g \subset [-R, R]$, then f is supported in $[-2R, 2R]$ and $|f(u)| \leq \|g\|_2^2$. Hence

$$\mathcal{B}_1(g) \geq -\left(C_0 + 2\sum_{n\leq e^{2R}} \frac{\Lambda(n)}{\sqrt{n}}\right)\|g\|_2^2 \tag{49}$$

$$\geq -C_1(1 + R)e^R\|g\|_2^2. \tag{50}$$

The last estimate follows already from $\Lambda(n) \leq \log n$ and comparison with an integral. It uses no prime number theorem.

Thus the positive Archimedean difference integral alone does not control the prime term with the needed uniformity. The elementary bound loses an exponential factor in the support radius. The subexponential replacement that would complete this route is GRH-strength by Corollary 4.3. The moment conditions do not repair that loss: the negative witnesses already satisfy them exactly.

4.4 Orthogonal compression does not supply the missing positivity

In the polynomially weighted space H_s, the adjoint of translation is

$$T_h^* v(x) = \frac{w_s(x-h)}{w_s(x)} v(x-h).$$

It is not T_{-h} when $s > 0$. Thus convolution by $g * g^*$, using the ordinary group involution, is generally not $A_g A_g^*$ in this Hilbert norm. The failure can produce a negative compressed trace even with the zeroth moment condition imposed.

Proposition 4.4 (A direct counterexample to automatic compression positivity)**.** *For every $s > 0$ there exist a rank-one orthogonal projection E on H_s and $g \in C_c^\infty(\mathbb{R})$ with $\int g = 0$ such that*

$$\operatorname{Tr}(E A_{g*g^*}) < 0.$$

Proof. Choose a nonzero real bump q near zero. For sufficiently large h, its translates near $-h, 0, h, 2h$ have disjoint supports. Set

$$A = \|q\|_{H_s}^2, \quad B = \|q(\cdot - h)\|_{H_s}^2, \quad u = q/\sqrt{A}, \quad v = q(\cdot - h)/\sqrt{B}, \quad e = (u+v)/\sqrt{2}.$$

The bump vectors u, v are orthonormal. With $r = \sqrt{B/A}$, a direct calculation gives

$$\langle (2I - T_h - T_{-h})e, e \rangle_{H_s} = 2 - \frac{1}{2}(r + r^{-1}).$$

Since $B/A \to \infty$ as $h \to \infty$, this number is negative for an appropriate fixed h.

Let ν_ε be a real smooth compactly supported approximate identity of integral one and put $g_\varepsilon = \nu_\varepsilon - \nu_\varepsilon(\cdot - h)$. Its integral is zero. Convolution by $g_\varepsilon * g_\varepsilon^*$ converges strongly on H_s to $2I - T_h - T_{-h}$. Therefore the displayed quadratic value remains negative for sufficiently small ε. Taking E to be projection onto the unit vector e gives the stated trace. This operator is rank one, so trace existence causes no difficulty. □

A true unitary representation would give a different conclusion. If the full arithmetic zero functional could be identified with

$$\mathcal{B}_\psi(g) = \operatorname{Tr}(U(g) P U(g)^*), \qquad P \geq 0,$$

with the positive operator trace class, it would be nonnegative and Theorem 4.2 would prove GRH. The equality with the full arithmetic functional is essential. Neither orthogonality of a projector nor nuclearity of a weighted convolution establishes it. Also, $(g * g^*)(0) = \|g\|_2^2 > 0$ for $g \neq 0$, so the hypothesis $\varphi(C_F^1) = 0$ in manuscript Theorem 12.5.1.1 excludes these convolution squares. It does not itself give a positivity assertion for their canonical extension.

Meyer's unconditional all-zero representation does not supply this unitarity; see the Introduction of *On a representation of the idele class group related to primes and zeros of L-functions*. Unitarizing a particular zero-jet quotient would also have to eliminate its nontrivial Jordan blocks at multiple zeros. That is stronger than a statement about zero locations; the trace-based positivity criterion itself allows arbitrary multiplicities on the critical line.

There is an explicit unitary modification of the weighted action:

$$(U_t f)(x) = \sqrt{\frac{w_s(x+t)}{w_s(x)}}\, f(x+t).$$

It restores the group involution, but changes the orbital operators. Indeed $Jf = w_s^{1/2} f$ is unitary from H_s to unweighted $L^2(\mathbb{R})$ and $JU_tJ^{-1} = T_t$. On the latter space every finite product $\prod_j (I - \lambda_j T_{h_j})$, $|\lambda_j| = 1$, has dense range: its Fourier multiplier is nonzero almost everywhere, so the kernel of its adjoint is zero. The corresponding closed finite-product quotients for the modified action are therefore zero. This modification does not preserve the earlier orbital quotients.

Similarly, a full spectral module with continuous zero functionals $\ell_\lambda(U_t v) = e^{\lambda t}\ell_\lambda(v)$ and bounds $\|U_t\| \le C_\varepsilon e^{\varepsilon|t|}$ for every $\varepsilon > 0$ would force $\operatorname{Re}\lambda = 0$ by taking $t \to \pm\infty$. The established strip norm gives only $e^{\alpha|t|}$ times a polynomial, which forces at most $|\operatorname{Re}\lambda| \le \alpha$. Changing the completion as $\alpha \downarrow 0$ does not prove such a bound on one module retaining all zeros.

A unitary dilation of a semigroup alone is weaker. For example, on $L^2(\mathbb{R}, a(\pi(a^2+\xi^2))^{-1}d\xi)$ let $V_t f(\xi) = e^{it\xi} f(\xi)$ and let P project onto the constant unit vector. The elementary Cauchy Fourier integral gives $PV_tP = e^{-a|t|}P$. For $t \ge 0$ this compresses a unitary group to a semigroup with generator $-a$, with $a > 0$. Thus such a dilation does not by itself put the compressed generator's spectrum on the imaginary axis; the full spectral identification remains essential.

4.5 What the recent semilocal route still needs

Connes–Consani's Theorem 1 proves an Archimedean comparison with a positive Sonin-space trace for tests supported in its stated small interval and satisfying its vanishing conditions. It is not a uniform positivity theorem after all primes allowed by an arbitrarily large support are included.

Connes' 2026 survey *The Riemann Hypothesis: Past, Present and a Letter Through Time*, Theorem 6.1, gives a real-zero conclusion under hypotheses on a simple, isolated lowest eigenvalue with an even eigenvector. Section 6.6 explicitly leaves that hypothesis for the Weil operator and the needed approximation of its eigenvector unresolved. Fact 6.4 gives strip convergence for the auxiliary k_λ, not the required eigenvector θ_{λ^2}.

Here is a precise sufficient approximation estimate, independently of how those functions are constructed. Write $\Theta_\lambda(u) = \theta_{\lambda^2}(e^u)$ and $K_\lambda(u) = k_\lambda(e^u)$, after fixing compatible nonzero scalar normalizations of the eigenvectors. If for every $0 < b < \frac{1}{2}$ one proved

$$\int_{\mathbb{R}} |\Theta_\lambda(u) - K_\lambda(u)| e^{b|u|}\, du \longrightarrow 0, \tag{51}$$

then their Fourier transforms would converge uniformly on $|\operatorname{Im} z| \le b$. Together with the survey's auxiliary convergence and the real-zero conclusion for Θ_λ, this would prove RH: any nonreal zero of Ξ lies in one of those substrips, and a small disk around it disjoint from the real axis would contradict Hurwitz's theorem (equivalently, Rouché's theorem).

The weight in (51) cannot be dropped. For a nonzero nonnegative $\phi \in C_c^\infty((-1,0))$ and $L = \log\lambda$, put

$$d_\lambda(u) = e^{-bL}(\phi(u-L) + \phi(-u-L)).$$

This error is real and even. Its unweighted L^1 and L^2 norms tend to zero, and its Fourier transform converges uniformly to zero on the real axis. Nevertheless, with Fourier convention $\widehat{d}(z) = \int d(u)e^{-izu}\,du$,

$$\widehat{d}_\lambda(ib) = \int_{-1}^{0} \phi(v)e^{bv}\,dv + e^{-2bL}\int_{-1}^{0}\phi(v)e^{-bv}\,dv \longrightarrow \int_{-1}^{0}\phi(v)e^{bv}\,dv > 0$$

does not tend to zero. This is a concrete escaping-boundary error. On a common logarithmic support $[-\log\lambda, \log\lambda]$, Cauchy–Schwarz would instead make the stronger rate $\|\Theta_\lambda - K_\lambda\|_2 = o(\lambda^{-b})$ sufficient for

the weighted estimate. Neither numerical agreement at finitely many zeros nor unweighted convergence supplies that rate.

The attempted completion therefore produces an explicit negative witness, an exact arithmetic quadratic form, and two precise sufficient estimates. It does not prove either missing global estimate. The weighted-projection shortcut is refuted by the example above; the cited semilocal and approximation results do not supply the stronger arithmetic comparison. No contradiction to GRH is inferred.

5 The mollifier norm and the zero-free boundary

The subsequent attempts replaced the unknown global comparison by explicit arithmetic norms. This section records the reduction and the cutoff tests. All assertions here concern the Riemann zeta function. A criterion equivalent to RH is not an independent proof of the estimate appearing in that criterion.

5.1 Definitions, Mellin identity, and the prime-counting connection

For an integer $N \geq 2$, define

$$a_{N,n} = \mu(n)\left(1 - \frac{\log n}{\log N}\right), \qquad V_N(s) = \sum_{n \leq N} a_{N,n} n^{-s}, \qquad Q_N(y) = \sum_{n \leq N} a_{N,n}\{y/n\}.$$

These logarithmically tapered coefficients are the classical Selberg approximation coefficients; see Báez-Duarte's paper listed in the sources. Set

$$\mathcal{N}_N = \int_0^\infty |Q_N(y)|^2 \frac{dy}{y^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|\zeta(\frac{1}{2}+it)V_N(\frac{1}{2}+it)|^2}{\frac{1}{4}+t^2}\,dt. \tag{52}$$

The second equality is Mellin Plancherel, since

$$\int_0^\infty \{y/n\} y^{-s-1}\,dy = -\frac{\zeta(s)}{s} n^{-s}, \qquad 0 < \Re s < 1.$$

The alternative denominator $1+t^2$ used in the discussion gives a norm S_N satisfying $S_N \leq \mathcal{N}_N \leq 4S_N$.

Let $m_N = V_N(1)$. For $0 < y < 1$, $Q_N(y) = m_N y$. For $1 \leq y \leq N$, the identities $\sum_{d|k} \mu(d) = \mathbf{1}_{\{k=1\}}$ and $\sum_{d|k} \mu(d)\log d = -\Lambda(k)$ give the exact formula

$$Q_N(y) = m_N y - 1 - \frac{\psi(y)}{\log N}. \tag{53}$$

Thus, with $D(y) = \psi(2y)/(2y) - \psi(y)/y$,

$$D(y) = \log N\left(\frac{Q_N(y)}{y} - \frac{Q_N(2y)}{2y} + \frac{1}{2y}\right) \qquad (1 \leq y \leq N/2).$$

In particular,

$$\int_1^{N/2} |D(y)|^2 dy \ll (\log N)^2(\mathcal{N}_N + 1). \tag{54}$$

This is a direct connection to the prime-energy criterion in the next section.

5.2 The exact power exponent of the norm

Put

$$\beta_* = \sup\{\Re\rho : \zeta(\rho) = 0,\ 0 < \Re\rho < 1\}.$$

No assumption that the supremum is attained is made.

Theorem 5.1 (Norm growth and the zero-free boundary)**.** *One has*

$$\limsup_{N\to\infty} \frac{\log \mathcal{N}_N}{\log N} = 2\beta_* - 1. \tag{55}$$

Consequently,

$$\big(\forall \varepsilon > 0,\ \mathcal{N}_N \ll_\varepsilon N^\varepsilon\big) \quad \Longleftrightarrow \quad \text{RH}.$$

The equivalence does not establish either side unconditionally.

Proof. First suppose $\mathcal{N}_N = O(N^{2\eta})$, with $\eta \geq 0$. For a fixed real $\sigma \in (1/2, 1)$, the Mellin formula and Cauchy–Schwarz on $(0, 1)$ and $[1, \infty)$ give

$$|V_N(\sigma)| \leq \frac{\sigma}{|\zeta(\sigma)|} \Big((1-\sigma)^{-2} + (2\sigma-1)^{-1}\Big)^{1/2} \sqrt{\mathcal{N}_N}.$$

Here $\zeta(\sigma) \neq 0$ on this real interval. Define

$$H_\sigma(u) = \sum_{n\geq 1} \mu(n) n^{-\sigma} (u - \log n)_+.$$

At $u = \log N$ it equals $(\log N)V_N(\sigma)$, and it is linear between successive such points. Hence $H_\sigma(u) = O_\sigma((1+u)e^{\eta u})$. Initially for $\Re z > 1 - \sigma$, termwise Laplace integration gives

$$\int_0^\infty e^{-zu} H_\sigma(u)\, du = \frac{1}{z^2 \zeta(\sigma + z)}.$$

The growth bound makes the left side holomorphic for $\Re z > \eta$. A zero of $\zeta(\sigma + z)$ in this half-plane would give a pole on the right, and $z = 0$ is outside this open half-plane. Therefore $\beta_* \leq \sigma + \eta$, and letting $\sigma \downarrow 1/2$ gives $\beta_* \leq 1/2 + \eta$.

For the reverse estimate, assume ζ has no zeros in $\Re s > \theta$, where $1/2 \leq \theta < 1$, and fix $0 < \delta < 1 - \theta$. The Balazard–Saias lemma, quoted as Lemma 2.1 by Báez-Duarte, implies on every fixed interior margin

$$\frac{1}{\zeta(\sigma + it)} \ll_{\theta,\delta,\nu} (1 + |t|)^\nu, \qquad \sigma \geq \theta + \delta, \quad \nu > 0.$$

At $\sigma = 1$, that lemma also bounds the partial sums of $\sum \mu(n) n^{-1-it}$ uniformly by $O((1 + |t|)^\nu)$. Abel summation extends the same bound uniformly to $\sigma > 1$. Logarithmic Perron inversion gives

$$V_N(\tfrac{1}{2} + it) = \frac{1}{2\pi i \log N} \int_{(c)} \frac{N^z}{z^2 \zeta(\frac{1}{2} + it + z)}\, dz, \qquad c > 1/2.$$

Shift to the fixed line $\Re z = a = \theta + \delta - 1/2 > 0$. No pole is crossed. The reciprocal estimate with $\nu < 1$ controls the horizontal segments and the new vertical integral, yielding

$$|V_N(\tfrac{1}{2} + it)| \ll_{\theta,\delta,\nu} \frac{N^a}{\log N} (1 + |t|)^\nu.$$

The unconditional second-moment estimate $\int_0^T |\zeta(1/2+it)|^2 dt \ll T\log(2T)$ makes the weighted integral in (52) finite if $\nu < 1/2$. Thus, for every fixed $\varepsilon > 0$,

$$\mathcal{N}_N \ll_{\theta,\varepsilon} \frac{N^{2\theta-1+\varepsilon}}{\log^2 N}. \tag{56}$$

The constants are not asserted to be uniform as the margin δ tends to zero.

For the endpoints, Minkowski's inequality and $\|\{y/n\}\|_{L^2(dy/y^2)} \asymp n^{-1/2}$ give $\mathcal{N}_N \ll N/\log^2 N$. On $(0,1)$ and $(1,2)$, respectively, the functions are $m_N y$ and $m_N y - 1$. Consequently

$$\mathcal{N}_N \geq 2m_N^2 - 2m_N \log 2 + \frac{1}{2} \geq \frac{1-(\log 2)^2}{2} > 0.$$

These bounds, the first implication, and (56) prove (55), including $\beta_* = 1/2$ and $\beta_* = 1$. □

5.3 The tails that can be controlled

Since $|\{y/n\}| \leq 1$ and $\sum_{n\leq N} \log(N/n) \leq N-1$,

$$|Q_N(y)| \leq \frac{N-1}{\log N}, \qquad \int_{N^2}^{\infty} |Q_N(y)|^2 \frac{dy}{y^2} \leq \frac{1}{\log^2 N}.$$

Also,

$$\int_0^1 |Q_N(y)|^2 \frac{dy}{y^2} = |m_N|^2 \leq (1+\log N)^2.$$

These estimates leave the interval $[1, N^2]$. Because each contribution to the squared norm is nonnegative, a controlled far tail cannot cancel an excessive contribution from the remaining interval.

5.4 Smoothing and averaging tests

Fixed higher-order taper. Replacing the weight by $(1-\log n/\log N)^k$ for a fixed positive integer k changes the logarithmic denominators but does not remove the zero obstruction. Indeed,

$$\int_0^{\infty} e^{-zu} \sum_n \mu(n) n^{-\sigma} (u - \log n)_+^k \, du = \frac{k!}{z^{k+1}\zeta(\sigma+z)}$$

in the initial convergence half-plane. The same fixed-margin argument has the power exponent $2\beta_* - 1$.

Taper order growing with the cutoff. For $k_N = \lceil \log N \rceil$, the weights satisfy $(1-\log n/\log N)^{k_N} \leq 1/n$, giving a bounded norm by $\sum n^{-3/2} < \infty$. But for every fixed n the coefficient tends to $\mu(n)/n$, and the reciprocal series is shifted to $1/\zeta(s+1)$. This is a different approximation problem. If instead $k_N = o(\log N)$, the fixed coefficients are preserved, but the absolute-value sum

$$B_N = \sum_{n\leq N} n^{-1/2} (1-\log n/\log N)^{k_N}$$

satisfies $\log B_N/\log N \to 1/2$. To see the lower bound, restrict to $n \leq N^{1-\delta}$ for fixed $\delta > 0$; the weight is at least $\delta^{k_N} = N^{o(1)}$. The upper bound is $B_N \leq 2\sqrt{N}$. Thus this absolute-value argument still costs $N^{1+o(1)}$ after squaring.

Averaging in the logarithmic cutoff. For real $X > 1$ use the same definitions with $n \le X$. Hilbert-valued Laplace Plancherel gives, for $a > 1/2$,

$$\int_1^\infty (\log X)^2 \mathcal{N}_X X^{-2a} \frac{dX}{X} = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \frac{|\zeta(\frac{1}{2}+it)|^2 \, dt\, dv}{(\frac{1}{4}+t^2)(a^2+v^2)^2 |\zeta(\frac{1}{2}+a+i(t+v))|^2}. \tag{57}$$

The initial restriction $a > 1/2$ puts the reciprocal in its absolute convergence region. Moving this identity to every $a > 0$ requires control of the possible intervening poles. Integrability of a proposed boundary quotient alone does not justify that continuation.

5.5 A prime-factor sign shortcut is false

For squarefree P set

$$F_P(y) = \sum_{d \mid P} \mu(d)\{y/d\} = \frac{\varphi(P)}{P} y - \#\{n \le y : (n, P) = 1\}.$$

If $p \nmid P$, then $F_{Pp}(y) = F_P(y) - F_P(y/p)$, whence

$$\|F_{Pp}\|^2 = (1 + p^{-1})\|F_P\|^2 - 2\int_0^\infty F_P(y) F_P(y/p) \frac{dy}{y^2}. \tag{58}$$

The cross-term need not be nonnegative. The actual pair $P = 30$, $p = 7$ gives

$$\int_0^\infty F_{30}(y) F_{30}(y/7) \frac{dy}{y^2} < -\frac{7}{1000}. \tag{59}$$

Here is a finite rational certificate. Put $\alpha = 4/15$, $c_j = \#\{1 \le n \le j : (n, 30) = 1\}$, and $e_j = \#\{1 \le n \le \lfloor j/7 \rfloor : (n, 30) = 1\}$. The function F_{30} is periodic and $|F_{30}| \le 14/15$, as checked on the thirty unit intervals of one period. Integrating on $[0, 210]$ and using $\log((j+1)/j) \ge 2/(2j+1) + 2/[3(2j+1)^3]$ gives the upper bound

$$U = 30\alpha^2 + \frac{(14/15)^2}{210} + \sum_{j=1}^{209} \left[\frac{c_j e_j}{j(j+1)} - \alpha(e_j + c_j/7) \left(\frac{2}{2j+1} + \frac{2}{3(2j+1)^3} \right) \right] < -\frac{7}{1000}.$$

All terms in U are rational. The positive tail bound and the direction of the logarithm inequality make this an upper, not a lower, certificate. This refutes the proposed prime-by-prime sign argument; it does not refute a global cancellation bound or RH.

5.6 What the external fixed half-plane input supplies

The 30 September 2026 manuscript *The Quasi-Riemann Hypothesis* in the OpenAI mathematics collection reports zero-freeness in $\Re s > 7/8$ for Dirichlet L-functions, including ζ. We checked the relevant statement and exponent comparison, not its full proof or the companion formal verification. Treating that reported theorem as an external input, (56) supplies

$$\mathcal{N}_N \ll_\varepsilon N^{3/4+\varepsilon} / \log^2 N.$$

It does not supply the subpower bound in Theorem 5.1. The inspected probe has a low estimate with exponent $3/16$ and a signal exponent $s - 11/16$, leaving the margin $s - 7/8$. The invoked plain-polynomial lemma is stated only for $\kappa \in [3/4, 1]$. Repeating that comparison below its stated range is not a valid iteration to $1/2$.

The same collection's ordinary two-point correlation result concerns fixed affine forms, with a power-of-logarithm saving for Liouville and a qualitative multiplicative-function conclusion. Such a fixed-form statement does not provide the uniform, growing-family estimate needed to sum the signed correlations in the inverse norm. No such uniform estimate was derived in these attempts.

Other tests retained as negative results. The broad-Gaussian Weil tests and finite Euler–Maclaurin approximants were also examined. Small energy or small Hardy-space error for those particular tests was not a sign theorem for every admissible test, nor a uniform nonvanishing theorem up to the critical boundary. Those computations supplied no estimate beyond the proved criteria recorded here. In particular, neither finite numerical verification nor an unspecified interchange of cutoff and boundary limits is used to close the argument.

6 Prime-counting energy and the exact remaining estimate

Status. The identities and equivalences in this section are proved. The subpower estimate stated below is not proved unconditionally. This is the Riemann-zeta, rational trivial-character specialization; none of the results in this section is asserted to complete GRH.

The purpose of the following calculations is to specify the missing arithmetic estimate without a positivity or probabilistic assumption hidden inside it. In particular, averaging the square does not make this estimate weaker than RH. A single off-line zero, of any multiplicity and at any height, is still detected by its Mellin transform.

6.1 Definitions and the subpower target

Use the right-continuous prime-counting convention

$$\Lambda(n) = \begin{cases} \log p, & n = p^k \text{ for a prime } p \text{ and } k \geq 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \psi(x) = \sum_{n \leq x} \Lambda(n).$$

For $x \geq 1$ put

$$E(x) = \psi(x) - x, \qquad r(x) = \frac{E(x)}{x}, \qquad D(x) = r(2x) - r(x) = \frac{\psi(2x)}{2x} - \frac{\psi(x)}{x}. \tag{60}$$

The compact correction arising in the explicit Möbius inversion is

$$h(x) = \mathbf{1}_{[1,2)}(x), \qquad d(x) = D(x) - h(x).$$

Values at isolated endpoints do not affect any Lebesgue integral. Define

$$\mathcal{J}_d(X) = \int_1^X |d(x)|^2 \, dx, \qquad \boxed{\mathcal{J}_d(X) \ll_\varepsilon X^\varepsilon \quad \text{for every } \varepsilon > 0.} \tag{61}$$

Each implied constant is allowed to depend on the fixed ε but not on X. The condition concerns all sufficiently large real X.

Theorem 6.1 (The averaged target is equivalent to RH)**.** *For any fixed compactly supported $h \in L^2([1,\infty), dx)$, let $d = D - h$. Then (61) holds if and only if RH holds. In particular, the assertion applies to $h = \mathbf{1}_{[1,2)}$.*

Proof. First assume (61). Consider the Mellin transform

$$\widehat{d}(s) = \int_1^\infty d(x) x^{-s}\, dx.$$

We verify its holomorphy, including the uniformity needed for complex zeros. Let K be a compact subset of $\{\Re s > 1/2\}$ and set $\sigma_K = \min_{s\in K} \Re s$. Choose $0 < \varepsilon < 2\sigma_K - 1$. Cauchy–Schwarz on a dyadic interval gives, uniformly for $s \in K$,

$$\begin{aligned}\int_{2^k}^{2^{k+1}} |d(x)x^{-s}|\, dx &\le \left(\int_{2^k}^{2^{k+1}} |d(x)|^2\, dx\right)^{1/2} \left(\int_{2^k}^{2^{k+1}} x^{-2\sigma_K}\, dx\right)^{1/2} \\ &\ll_{K,\varepsilon} 2^{-k(\sigma_K - 1/2 - \varepsilon/2)}.\end{aligned}$$

The right side is summable. Factors $(\log x)^j$ introduce only a polynomial factor in k, which remains summable. The integral therefore converges absolutely and locally uniformly, and differentiation under the integral is valid. Consequently

$$\widehat{d} \text{ is holomorphic throughout } \Re s > \tfrac{1}{2}. \tag{62}$$

For $\Re s > 1$, absolute convergence of the von Mangoldt Dirichlet series gives

$$\int_1^\infty \frac{\psi(x)}{x} x^{-s}\, dx = \frac{1}{s}\sum_{n\ge 1} \frac{\Lambda(n)}{n^s} = -\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)}.$$

In the dilated integral, the substitution $y = 2x$ contributes 2^{s-1}. There is no omitted initial integral because $\psi(y) = 0$ for $1 \le y < 2$. It follows that

$$\widehat{d}(s) = -\frac{2^{s-1}-1}{s}\frac{\zeta'(s)}{\zeta(s)} - H(s), \qquad H(s) = \int_1^\infty h(x) x^{-s}\, dx. \tag{63}$$

The function H is entire: its integration interval is compact and h is integrable there. For the indicated correction,

$$H(s) = \frac{1 - 2^{1-s}}{s-1}, \qquad H(1) = \log 2,$$

with the singularity at one removed. The apparent pole at $s = 1$ in the first term of (63) is likewise removable, because $2^{s-1} - 1$ vanishes there.

Suppose that ρ is a nontrivial zero of ζ of multiplicity $m \ge 1$ with $\Re\rho > 1/2$. Locally,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{m}{s-\rho} + \text{a holomorphic function.}$$

The meromorphic right side of (63) has at $s = \rho$ the residue

$$-\frac{m(2^{\rho-1}-1)}{\rho} \neq 0. \tag{64}$$

Indeed $\rho \neq 0$ and $|2^{\rho-1}| = 2^{\Re\rho - 1} < 1$ for a nontrivial zero. The entire compact correction cannot change this residue. Uniqueness of meromorphic continuation from $\Re s > 1$ now contradicts (62), since

$\Re\rho > 1/2$. There are no nontrivial zeros to the right of $1/2$. The functional-equation symmetry then places every nontrivial zero on the critical line. This proves the forward implication, including complex zeros and arbitrary multiplicities.

Conversely, under RH the standard prime-counting estimate is

$$E(x) \ll x^{1/2} \log^2(2x). \tag{65}$$

This estimate is explicitly recorded as RH conditional in the introduction, equation (1), of Brent, Platt and Trudgian's paper on the mean square of the prime-number-theorem error.[2] Equation (60) yields $D(x) \ll x^{-1/2} \log^2(2x)$. The compact correction has finite squared norm, so

$$\mathcal{J}_d(X) \ll_h 1 + \int_2^X \frac{\log^4(2x)}{x}\, dx \ll_h 1 + \log^5(2X).$$

Every fixed power of $\log(2X)$ is $O_\varepsilon(X^\varepsilon)$. This proves the converse. □

Remark 6.2 (What the equivalence does not establish). The theorem proves a logical equivalence, not the estimate (61) unconditionally. Aggregation in the squared integral cannot hide an off-line zero: the locally uniform Mellin argument would exclude each of its nonzero residues separately. Zeros on the critical line lie on the boundary of the holomorphy region and are compatible with the logarithmic growth in the converse.

6.2 An exact jump-energy identity

Return to the individual error $r(x) = E(x)/x$. Put

$$\begin{aligned} R(X) &= \int_1^X r(y)^2\, dy, \\ C(X) &= \int_{(1,X]} r(y-)\big(d\psi(y) - dy\big) = \sum_{2\le n\le X} \Lambda(n) r(n-) - \int_1^X r(y)\, dy, \\ Q(X) &= \sum_{n\le X} \frac{\Lambda(n)^2}{n}. \end{aligned} \tag{66}$$

The notation $r(y-)$ specifies the left limit at a jump. In the continuous integral its value at isolated points is immaterial. No probability measure is involved in the definition of C.

Proposition 6.3 (Exact energy balance). *For every real $X \ge 1$, with the right-continuous endpoint convention,*

$$\boxed{R(X) + \frac{E(X)^2}{X} = 1 + Q(X) + 2C(X).} \tag{67}$$

Unconditionally,

$$Q(X) \ll \log^2(2X), \qquad C(X) \ge -\frac{1 + Q(X)}{2}. \tag{68}$$

Proof. Between jumps, $E'(y) = -1$, and hence

$$\frac{d}{dy}\frac{E(y)^2}{y} = -2r(y) - r(y)^2.$$

[2] R. P. Brent, D. J. Platt and T. S. Trudgian, *The mean square of the error term in the prime number theorem* (2020), https://arxiv.org/pdf/2008.06140. Its mean-square upper bound is also RH conditional; its stated lower bound is unconditional.

At an integer $n \geq 2$, the jump is $\Delta E(n) = \Lambda(n)$, giving

$$\Delta\left(\frac{E(n)^2}{n}\right) = 2\Lambda(n)r(n-) + \frac{\Lambda(n)^2}{n}.$$

Integration over $(1, X]$, summation of these jumps, and $E(1) = -1$ prove (67).

The elementary Chebyshev bound $\psi(x) \ll x$ gives by partial summation

$$\sum_{n \leq X} \frac{\Lambda(n)}{n} = \frac{\psi(X)}{X} + \int_1^X \frac{\psi(t)}{t^2}\, dt \ll \log(2X).$$

Since $\Lambda(n)^2 \leq \Lambda(n) \log n$, the first assertion of (68) follows. The second follows from the nonnegativity of the left side of (67). □

Corollary 6.4 (The one-sided covariance target)**.** *The following condition is equivalent to RH:*

$$\text{for every } \varepsilon > 0 \text{ there is } K_\varepsilon < \infty \text{ such that } C(X) \leq K_\varepsilon X^\varepsilon \quad (X \geq 2). \tag{69}$$

Only an upper bound is required here.

Proof. Assume (69). Dropping $R(X) \geq 0$ from (67), and using (68), shows that for every $\delta > 0$

$$E(X) \ll_\delta X^{1/2+\delta}.$$

Indeed use (69) with $\varepsilon = 2\delta$ and absorb the logarithm into the same positive power. Therefore $D(x) \ll_\delta x^{-1/2+\delta}$, and

$$\int_1^X |D(x) - h(x)|^2\, dx \ll_{\delta,h} 1 + X^{2\delta}.$$

As δ is arbitrary, Theorem 6.1 gives RH.

Under RH, (65) gives $R(X) \ll \log^5(2X)$ and $E(X)^2/X \ll \log^4(2X)$. The exact balance and the bound on Q imply $|C(X)| \ll \log^5(2X)$, proving the converse. □

6.3 Arithmetic centering and the available Cauchy–Schwarz bound

For an integer cutoff $N \geq 2$, abbreviate

$$r_n = \frac{\psi(n-)}{n} - 1,$$

and define the nonnegative discretization correction

$$B_{\mathrm{disc}}(N) = \sum_{n=2}^{N} \psi(n-) \left(\log \frac{n}{n-1} - \frac{1}{n}\right). \tag{70}$$

Then the exact centered form is

$$\boxed{C(N) = \sum_{n=2}^{N} (\Lambda(n) - 1) r_n - B_{\mathrm{disc}}(N), \qquad 0 \leq B_{\mathrm{disc}}(N) \ll \log(2N).} \tag{71}$$

To check it, note that $\psi(y) = \psi(n-)$ on $(n-1, n)$, so

$$\int_{n-1}^{n} r(y)\, dy = \psi(n-) \log \frac{n}{n-1} - 1.$$

Summing and subtracting $\sum r_n$ gives exactly $B_{\mathrm{disc}}(N)$. Its sign and size follow from

$$0 < \log\frac{n}{n-1} - \frac{1}{n} = \sum_{j\ge 2}\frac{1}{jn^j} \le \frac{1}{2n(n-1)},$$

together with $\psi(n-) \ll n$.

This centering is rigorous, but it does not establish cancellation in the remaining weighted sum. The estimates it supplies directly are as follows. For $y \in (n-1, n)$,

$$|r(y) - r_n| = \psi(n-)\left(\frac{1}{y} - \frac{1}{n}\right) \ll \frac{1}{n}.$$

Squaring, integrating over these unit intervals, and summing proves

$$\sum_{n=2}^{N} r_n^2 \le 2R(N) + O(1). \tag{72}$$

Moreover,

$$\sum_{n=2}^{N}(\Lambda(n) - 1)^2 \ll N\log(2N),$$

since $\sum_{n\le N}\Lambda(n)^2 \le \log N\,\psi(N)$. Dropping the nonnegative B_{disc} in (71) and applying Cauchy–Schwarz therefore gives

$$C(N) \ll \sqrt{N\log(2N)}\sqrt{R(N)+1}.$$

Young's inequality yields, for every fixed $\kappa > 0$,

$$\boxed{C(N) \le \kappa R(N) + O_\kappa(N\log(2N)).} \tag{73}$$

All these estimates are unconditional. They do not imply (69). For $0 < \kappa < 1/2$, substitution into the exact energy balance allows the $2\kappa R(N)$ term to be absorbed, but leaves

$$(1-2\kappa)R(N) + \frac{E(N)^2}{N} \ll_\kappa N\log(2N).$$

The remaining error is far larger than the required subpower scale. Absorbing a fraction of the unknown energy does not remove this error.

Remark 6.5 (The missing martingale assertion). The measure $d\psi - dy$ is deterministic. Subtracting dy centers its main term but does not make its increments conditionally mean zero, and $r(y-)$ is itself determined by preceding prime-counting values. There is no established orthogonality that makes $C(X)$ vanish or have a sufficiently small upper bound. The nonnegative jump term $Q(X)$ controls the full energy only after the signed covariance $C(X)$ has also been bounded. Dropping C, or replacing it by a martingale expectation without a justified probabilistic model and transfer theorem, would assume the missing step.

6.4 The measure in a dilation argument matters

Under $y = e^t$, the requested norm is

$$\int_1^X |d(y)|^2\,dy = \int_0^{\log X} e^t|d(e^t)|^2\,dt. \tag{74}$$

An unweighted logarithmic-coordinate estimate instead controls $\int |d(y)|^2\,dy/y$. These are different norms. For example, $d_0(y) = y^{-1/4}$ satisfies

$$\int_1^\infty |d_0(y)|^2 \frac{dy}{y} = 2, \qquad \int_1^X |d_0(y)|^2\,dy = 2(\sqrt{X} - 1).$$

Thus a bounded logarithmic norm alone cannot prove (61). A unitary logarithmic reformulation of the target must use $e^{t/2}d(e^t)$, retaining the corresponding weights in every dilation or averaging operator.

Unresolved point. No independent proof of (61) or (69) has been obtained. The exact identities locate the required cancellation, and the Mellin argument proves what that cancellation would imply. They do not provide the cancellation itself. The present writeup therefore remains a partial repair in the zeta specialization, with RH and GRH unproved.

6.5 The prime-pair kernel and cancellation of the density main term

The desired aggregate can also be written directly as a weighted sum over prime powers. For $X \geq 2$ define

$$w_n(y) = \mathbf{1}_{\{y \geq n/2\}} - 2\mathbf{1}_{\{y \geq n\}}, \qquad K_X(m,n) = \frac{1}{4}\int_X^{2X} w_m(y) w_n(y) \frac{dy}{y^2}.$$

Then

$$\int_X^{2X} D(y)^2 dy = \sum_{m,n \leq 4X} \Lambda(m)\Lambda(n) K_X(m,n). \tag{75}$$

This is a finite identity, obtained by squaring $D(y) = (2y)^{-1}\sum_{n \leq 4X} \Lambda(n) w_n(y)$. With

$$F_X(t) = \begin{cases} 1/(2X), & t \leq X, \\ 1/t - 1/(2X), & X < t < 2X, \\ 0, & t \geq 2X, \end{cases}$$

direct integration gives, for $m \leq n$,

$$K_X(m,n) = \frac{1}{4}\left[F_X(n/2) + 2F_X(n) - 2F_X\big(\max(m, n/2)\big)\right].$$

Although this is a positive semidefinite Gram matrix, its entries have both signs. Its diagonal contributes $O(\log X)$: indeed $K_X(n,n) = F_X(n/2)/4$, and $\sum_{n \leq Y} \Lambda(n)^2 \ll Y \log(2Y)$.

The cancellation of the continuous prime-density main term is exact:

$$\int_0^{4X} K_X(u,v)\,du = 0.$$

For each $y \in [X, 2X]$, the corresponding inner integral of $w_u(y)$ is $2y - 2y = 0$. Thus (75) can equivalently be written as a quadratic integral against $d\psi - du$ in both variables. That cancellation does not estimate the remaining discrepancy.

For clarity, let P_X and M_X be the sums of the positive and of the absolute values of the negative contributions in (75). On $m, n \leq X$, $K_X(m,n) = 1/(8X)$, so the prime number theorem gives

$$P_X \geq \frac{\psi(X)^2}{8X} = \frac{X}{8} + o(X).$$

On $m \le X$ and $2X < n \le 3X$, $K_X(m,n) = 1/(8X) - 1/(2n) \le -1/(24X)$. Using this rectangle and its transpose gives

$$M_X \ge \frac{\psi(X)(\psi(3X) - \psi(2X))}{12X} = \frac{X}{12} + o(X).$$

Since $|K_X| \le 1/(8X)$, $P_X + M_X \le \psi(4X)^2/(8X) = 2X + o(X)$. Hence both signed masses are of order X, whereas the target requires $P_X - M_X \ll_\varepsilon X^\varepsilon$. Bounds for only the positive pieces, or replacing a negative-weighted correlation by an upper bound, do not establish this difference.

6.6 The precise consequence of a fixed zero-free half-plane

Proposition 6.6 (Mean-square consequence of a zero bound)**.** *Suppose every nontrivial zero of ζ has real part at most θ, where $1/2 \le \theta < 1$. Then*

$$\int_X^{2X} |\psi(y) - y|^2 \, dy \ll X^{2\theta+1}.$$

Consequently,

$$\mathcal{J}_d(X) \ll_\theta \begin{cases} 1 + \log X, & \theta = 1/2, \\ X^{2\theta-1}, & \theta > 1/2. \end{cases} \tag{76}$$

This is an implication from the stated zero bound, not a new exclusion of zeros.

Proof. The usual symmetric-height explicit formula, away from the countable set of prime-power endpoints, represents $\psi(y) - y$ as $-\lim_{T\to\infty} \sum_{|\Im\rho|\le T} y^\rho/\rho$ plus a bounded elementary term on $y \ge 2$. We justify the square integral without assuming simplicity of the zeros. For any pair $\rho = \beta + i\gamma$, $\rho' = \beta' + i\gamma'$,

$$\left| \int_X^{2X} y^{\rho+\overline{\rho'}} dy \right| \ll \frac{X^{\beta+\beta'+1}}{1 + |\gamma - \gamma'|} \le \frac{X^{2\theta+1}}{1 + |\gamma - \gamma'|}.$$

The zero-counting bound in unit-height intervals is $O(\log(2 + |t|))$. It implies the unconditional convergence

$$\sum_{\rho,\rho'} \frac{1}{|\rho|\,|\rho'|\,(1 + |\gamma - \gamma'|)} < \infty.$$

For completeness, partition positive heights into unit intervals with indices k, ℓ. When $k/2 \le \ell \le 2k$, summing the reciprocal height difference contributes $O(\log k)$, so the total is bounded by $\sum_{k\ge1} O(\log^3(2+k)/(1+k)^2)$. When $\ell > 2k$, the difference is comparable to ℓ; summing $\log(2+k)/(1+k)$ over $k < \ell/2$ again leaves a bound of the same summable form. The other ranges, including opposite signs of the heights, are no larger; bounded heights contribute only finitely many zeros.

The finite zero sums are therefore Cauchy in $L^2([X, 2X])$ with squared norm $O(X^{2\theta+1})$. Their limit agrees almost everywhere with the explicit formula, proving the first assertion. Writing $r(y) = (\psi(y) - y)/y$ gives $\int_X^{2X} r(y)^2 dy \ll X^{2\theta-1}$. The inequality $|r(2y) - r(y)|^2 \le 2r(2y)^2 + 2r(y)^2$ gives the same dyadic bound for D. Summing dyadic intervals is a geometric sum when $\theta > 1/2$ and contributes $O(\log X)$ when $\theta = 1/2$. The fixed compact correction h changes this only by a bounded initial contribution. □

The RH endpoint agrees with the conditional Cramér-type mean-square bound discussed by Brent, Platt and Trudgian. Taking the reported 7/8 theorem described in Section 5 as an external input gives the concrete bound

$$\mathcal{J}_d(X) \ll X^{3/4}.$$

This retains a positive power. It neither proves (61) nor provides an iteration decreasing that power to zero.

7 Arithmetic inversion and the remaining signed aggregate

This section records the arithmetic inversion attempts in the rational zeta specialization. The factorial and divisor identities, the forcing estimates, and the finite quadratic decomposition below are proved. They do not establish the required subpower estimate. Put

$$\psi(x) = \sum_{n \le x} \Lambda(n), \qquad D(x) = \frac{\psi(2x)}{2x} - \frac{\psi(x)}{x}, \qquad h(x) = \mathbf{1}_{[1,2)}(x), \qquad d = D - h.$$

Extend these functions by zero on $0 < x < 1$; this agrees with the definition of D, since $\psi(t) = 0$ for $t < 2$. The unresolved estimate considered here is

$$J(X) := \int_1^X |d(x)|^2 \, dx \ll_\varepsilon X^\varepsilon \qquad \text{for every } \varepsilon > 0. \tag{77}$$

No result in this section proves (77).

7.1 An exact convolution with a small explicit forcing

Write

$$L(t) = \log(\lfloor t \rfloor !), \qquad H_k = \sum_{j=1}^{k} \frac{1}{j}, \quad H_0 = 0, \qquad F(t) = \frac{1}{2} L(2t) - L(t).$$

Here $0! = 1$. All floor functions take their ordinary right-continuous values, so the following finite identities also hold at their jump points.

Proposition 7.1 (Exact factorial forcing)**.** *For $x \ge 1$, define*

$$B(x) = \frac{L(2x)}{2x} - \frac{L(x)}{x}, \qquad b(x) = B(x) - H_{\lfloor x \rfloor} + H_{\lfloor x/2 \rfloor}.$$

Then

$$\sum_{n \le x} \frac{1}{n} D(x/n) = B(x), \tag{78}$$

$$d(x) + \sum_{2 \le n \le x} \frac{1}{n} d(x/n) = b(x), \tag{79}$$

$$d(x) = \sum_{n \le x} \frac{\mu(n)}{n} b(x/n), \tag{80}$$

$$D(x) = \frac{1}{x} \sum_{n \le x} \mu(n) F(x/n). \tag{81}$$

Moreover,

$$B(x) = \log 2 + O\Big(\frac{\log(2x)}{x}\Big), \qquad b(x) = O\Big(\frac{\log(2x)}{x}\Big).$$

The constants are absolute.

Proof. The identity $\sum_{r|k}\Lambda(r)=\log k$ gives

$$\sum_{n\ge1}\psi(x/n)=\sum_{k\le x}\log k=L(x).$$

Taking the normalized difference at $2x$ and x proves (78). Also,

$$\sum_{n\ge1}\frac{1}{n}h(x/n)=\sum_{x/2<n\le x}\frac{1}{n}=H_{\lfloor x\rfloor}-H_{\lfloor x/2\rfloor}.$$

Subtracting gives (79). The coefficient sequences $1/n$ and $\mu(n)/n$ are inverse under Dirichlet convolution, proving (80). Inverting the identity for L gives (81); terms with $x<n\le 2x$ in $L(2x/n)$ vanish because $\lfloor 2x/n\rfloor=1$. Finally, Stirling's formula and the harmonic-number estimate give

$$\frac{L(x)}{x}=\log x-1+O(\log(2x)/x),\qquad H_{\lfloor x\rfloor}-H_{\lfloor x/2\rfloor}=\log 2+O(1/x).$$

These prove the asserted forcing bounds, with the bounded initial interval absorbed into the constants. □

The smallness of b is an estimate for the forcing in (79). Estimating its inverse still requires control of the signed sum in (80).

7.2 Stirling expansion with the floor terms retained

Define the periodic functions

$$\delta(t)=\lfloor 2t\rfloor-2\lfloor t\rfloor,\qquad \sigma(t)=2\delta(t)-1,\qquad p_2(t)=\lfloor t\rfloor-2\lfloor t/2\rfloor.$$

Thus $\delta,r\in\{0,1\}$ and $\sigma\in\{-1,1\}$.

Lemma 7.2 (Uniform floor-preserving expansion)**.** *Uniformly for every real $t\ge1$,*

$$F(t)=t\log 2+\frac{\sigma(t)}{4}\log t-\frac{1}{4}\log\pi-\frac{1}{2}\{2t\}\log 2+O(t^{-1}).\tag{82}$$

Consequently,

$$b(t)=\frac{\sigma(t)\log t}{4t}+\frac{c(t)}{t}+O(t^{-2}),\qquad c(t)=\frac{1}{2}-p_2(t)-\frac{1}{4}\log\pi-\frac{1}{2}\{2t\}\log 2.\tag{83}$$

In particular, c is bounded and periodic of period 2.

Proof. Writing $\alpha=\{t\}$, the integer Stirling expansion, followed by Taylor expansion about the real variable t, gives

$$L(t)=t\log t-t+\left(\frac{1}{2}-\alpha\right)\log t+\frac{1}{2}\log(2\pi)+O(t^{-1}).$$

The standard gamma and digamma expansions used here are recorded in DLMF, Section 5.11. The remainder is uniform in $0\le\alpha<1$. For $t\ge2$ this follows from $\lfloor t\rfloor\ge t/2$; enlarging the constant covers $1\le t<2$. Since $\{2t\}=2\{t\}-\delta(t)$, subtraction in $\frac12L(2t)-L(t)$ gives the logarithmic coefficient $\delta(t)/2-1/4=\sigma(t)/4$. The remaining constant terms are

$$\left(\frac{1}{4}-\frac{1}{2}\{2t\}\right)\log 2-\frac{1}{4}\log(2\pi)=-\frac{1}{4}\log\pi-\frac{1}{2}\{2t\}\log 2.$$

This proves (82), including its values at integers.

Likewise, uniformly for $u \geq 1$,

$$H_{\lfloor u \rfloor} = \log u + \gamma + \frac{1/2 - \{u\}}{u} + O(u^{-2}).$$

For $t \geq 2$, apply this at t and $t/2$, using $2\{t/2\} = \{t\} + p_2(t)$, to obtain

$$H_{\lfloor t \rfloor} - H_{\lfloor t/2 \rfloor} = \log 2 + \frac{p_2(t) - 1/2}{t} + O(t^{-2}).$$

An enlarged uniform constant again covers $1 \leq t < 2$. Combining this formula with $b = F/t - H_{\lfloor t \rfloor} + H_{\lfloor t/2 \rfloor}$ proves (83). □

At an integer $t = k$, formula (82) reads

$$\frac{1}{2} \log \binom{2k}{k} = k \log 2 - \frac{1}{4} \log(\pi k) + O(k^{-1}).$$

At a half-integer the logarithmic coefficient has the opposite sign. Replacing this coefficient by its integer value on all real arguments would therefore omit a leading term.

Proposition 7.3 (Sharp leading forcing tail)**.** *The forcing belongs to $L^2(1, \infty)$, and, as $T \to \infty$,*

$$\int_T^\infty |b(t)|^2 \, dt = \frac{(\log T)^2}{16T} + O\left(\frac{\log T}{T}\right). \tag{84}$$

Proof. Since $\sigma(t)^2 = 1$, equation (83) gives

$$b(t)^2 = \frac{(\log t)^2}{16t^2} + O\left(\frac{\log(2t)}{t^2}\right).$$

Integrate, using

$$\int_T^\infty \frac{(\log t)^2}{t^2} \, dt = \frac{(\log T)^2 + 2 \log T + 2}{T}.$$

The function b is bounded on compact subintervals of $[1, \infty)$. □

7.3 The leading floor term reconstructs the discrepancy

The preceding expansion exposes the arithmetic term that a smooth Stirling approximation would lose.

Proposition 7.4 (Exact weighted floor identity)**.** *For $x \geq 1$, put*

$$U(x) = \sum_{n \leq x} \mu(n) \log(x/n), \qquad W(x) = \sum_{x < n \leq 2x} \mu(n) \log(x/n).$$

Then

$$\sum_{n \leq x} \mu(n) \sigma(x/n) \log(x/n) = 4xD(x) - 2 \log x - U(x) - 2W(x). \tag{85}$$

Proof. The divisor identity

$$\sum_{n|k} \mu(n) \log(x/n) = \log x \, \mathbf{1}_{\{k=1\}} + \Lambda(k)$$

gives the finite formulas

$$\sum_{n\le x} \mu(n) \log(x/n) \lfloor x/n \rfloor = \log x + \psi(x),$$
$$\sum_{n\le x} \mu(n) \log(x/n) \lfloor 2x/n \rfloor = \log x + \psi(2x) - W(x).$$

In the second formula, the omitted terms $x < n \le 2x$ have $\lfloor 2x/n \rfloor = 1$. Substitute

$$\sigma(x/n) = 2\lfloor 2x/n \rfloor - 4\lfloor x/n \rfloor - 1.$$

The resulting prime-counting term is $2\psi(2x) - 4\psi(x) = 4xD(x)$, proving the claim. □

Inserting (82) into (81) multiplies the left side of (85) by $1/(4x)$. It therefore returns $D(x)$ itself with coefficient one. This identity does not provide a smaller estimate for that term.

Nor does the uniform remainder in a fixed-order Stirling expansion become small merely by summing it. At the order used above, absolute summation gives

$$\frac{1}{x} \sum_{n\le x} O(n/x) = O(1).$$

Restricting to $n \le x/T$, with $1 \le T \le x$, improves this particular remainder to $O(T^{-2} + (xT)^{-1})$, but leaves the range $x/T < n \le x$ to be estimated. The original bound $|b(t)| \ll \log(2t)/t$ gives the useful uniform estimate

$$\left| \sum_{n\le x/T} \frac{\mu(n)}{n} b(x/n) \right| \ll \frac{\log(2T)}{T}.$$

These are bounds for specified parts of the inverse. They do not control the remaining signed aggregate.

7.4 What a fixed compact correction can change

For this subsection the Mellin convention is

$$\widehat{f}(s) = \int_1^\infty f(x) x^{-s} \, dx.$$

Directly from the Dirichlet series for $-\zeta'/\zeta$, initially for $\Re s > 1$,

$$\widehat{D}(s) = -k(s) \frac{\zeta'(s)}{\zeta(s)}, \qquad k(s) = \frac{2^{s-1} - 1}{s}. \tag{86}$$

Indeed, changing variables in the $2x$ term leaves no lower-end correction because $\psi(t) = 0$ on $1 \le t < 2$.

Proposition 7.5 (Fixed-support corrections)**.** *Let $q \in L^2(1, A)$, extended by zero outside $[1, A]$, and let $d_q = D - q$. For $X \ge A$, the difference*

$$\int_1^X |d_q(x)|^2 \, dx - \int_1^X |D(x)|^2 \, dx$$

is a finite constant independent of X. The meromorphic continuation of $\widehat{d}_q = \widehat{D} - \widehat{q}$ retains the pole at every nontrivial zero ρ of ζ, with residue

$$-m_\rho \frac{2^{\rho-1}-1}{\rho}.$$

Proof. The energy difference is supported on the fixed interval $[1, A]$, where D is bounded and q is square integrable. The transform $\widehat{q}$ is entire. At a zero of multiplicity m_ρ, ζ'/ζ has residue m_ρ, and $2^{\rho-1} - 1 \neq 0$ because the zeros of this factor have real part one. Equation (86) proves the pole statement. □

For the canonical correction $q = h$, the exact recurrence gives

$$\widehat{b}(s) = -k(s)\zeta'(s) - \zeta(s)\widehat{h}(s), \qquad \widehat{h}(s) = \int_1^2 x^{-s}\, dx.$$

The decay of b makes $\widehat{b}$ holomorphic for $\Re s > 0$. At a nontrivial zero of multiplicity m, however, its numerator vanishes to order exactly $m - 1$. Division by $\zeta(s)$ retains one pole. The forcing-tail estimate therefore does not justify a bounded inverse on the desired space. This observation concerns fixed compact corrections; it asserts no obstruction for every possible change of approximation.

7.5 The exact signed energy after gcd decomposition

For positive integers a, c, define

$$K_Y(a,c) = \begin{cases} \displaystyle\int_{\max(a,c)}^{Y} b(t/a)b(t/c)\, dt, & Y \geq \max(a,c), \\ 0, & Y < \max(a,c). \end{cases}$$

Proposition 7.6 (Finite signed quadratic form)**.** *For every $X \geq 1$,*

$$J(X) = \sum_{\substack{a,c \leq X \\ (a,c)=1}} \frac{\mu(a)\mu(c)}{ac} \sum_{\substack{g \leq X/\max(a,c) \\ (g,ac)=1}} \frac{\mu(g)^2}{g} K_{X/g}(a,c). \tag{87}$$

The diagonal in the original pair variables has size

$$\sum_{n \leq X} \frac{\mu(n)^2}{n} \int_1^{X/n} b(t)^2\, dt = \frac{6}{\pi^2}\|b\|_{L^2(1,\infty)}^2 \log X + O(1). \tag{88}$$

Absolute summation of the entire quadratic form gives the unconditional bound $J(X) \ll X$.

Proof. Expanding the finite inverse (80) gives

$$J(X) = \sum_{m,n \leq X} \frac{\mu(m)\mu(n)}{mn} \int_{\max(m,n)}^{X} b(x/m)b(x/n)\, dx.$$

Write $m = ga$, $n = gc$, $(a,c) = 1$, and substitute $x = gt$. The product of Möbius coefficients vanishes unless $(g, ac) = 1$; otherwise it equals $\mu(g)^2\mu(a)\mu(c)$. This proves (87).

For the diagonal, interchange the finite sum and integral and use

$$\sum_{n\le Y}\frac{\mu(n)^2}{n}=\frac{6}{\pi^2}\log Y+O(1).$$

This estimate follows from $\mu(n)^2=\sum_{r^2\mid n}\mu(r)$ and the harmonic-number estimate. Both $\int_1^\infty b(t)^2\,dt$ and $\int_1^\infty b(t)^2\log t\,dt$ converge by (83); its tail bound also controls truncation at $t=X$. This proves (88).

Finally, for $m\le n$ put $r=n/m$. The forcing estimate gives

$$\int_n^X |b(x/m)b(x/n)|\,dx\le n\int_1^\infty |b(rt)b(t)|\,dt\ll m(1+\log r),$$

because $\int_1^\infty |b(t)|(1+\log t)\,dt/t<\infty$. Thus absolute summation is at most a constant times

$$\sum_{n\le X}\frac{1}{n}\sum_{m\le n}(1+\log(n/m))\ll X,$$

as asserted. □

The common divisor in (87) contributes the nonnegative weight $\mu(g)^2/g$. The relevant signs remain in $\mu(a)\mu(c)$. In the balanced primitive range $X/2<a,c\le X$, the inner sum contains only $g=1$. Consequently, averaging common-divisor densities supplies no additional estimate for that range. The cutoff and the factorial-floor kernel are not multiplicative in a,c, so the remaining signed sum does not factor into an Euler product.

Equation (88) is a calculation of one part of the quadratic form; it does not establish that the diagonal is its leading term. The proved forcing decay, the exact floor cancellation, and the gcd decomposition leave the same signed aggregate uncontrolled. The estimate (77), and the corresponding all-character argument needed for GRH, remain unproved.

8 Further attempted closures and the estimates they do not supply

This section records further attempts to prove the arithmetic estimate. Each produces a valid identity or bound, but none supplies the missing subpower estimate. The countermodel below tests only a specified collection of weakened hypotheses; it is not a model of the full prime arithmetic and is not a counterexample to RH.

Attempt	**Proved input**	**Missing claim and scope**
Selberg identity	Exact identity for ζ''/ζ; the symmetry formula has an $O(x)$ remainder.	A subpower covariance estimate does not follow from that remainder, positivity, and PNT alone.
Completed ξ	Horizontal modulus monotonicity in a zero-free half-plane; the shifted comparison has an exact zero-boundary threshold.	Extending the comparison to arbitrarily small shifts requires a new zero-free estimate.
Dilation Gram matrix	Exact quadratic form for the actual forcing; separated dilations have decaying correlations.	Nearby logarithmic shifts have a density loss. The required cancellation for the actual Möbius coefficients is unproved.
Dyadic difference	A bounded inverse for the weighted dilation difference, whenever its input has finite norm.	Finite energy in the required measure is still an assumption; the logarithmic measure gives a weaker statement.

8.1 What the Selberg symmetry formula cancels

Write $L(s) = -\zeta'(s)/\zeta(s)$, initially for $\Re s > 1$. Absolute convergence gives

$$\sum_{n\geq 1} \frac{\Lambda(n)\log n}{n^s} + \sum_{n\geq 1} \frac{(\Lambda * \Lambda)(n)}{n^s} = -L'(s) + L(s)^2 = \frac{\zeta''(s)}{\zeta(s)}. \tag{89}$$

The standard von Mangoldt form of Selberg's symmetry formula is

$$S_\Lambda(x) := \sum_{n\leq x} \Lambda(n)\log n + \sum_{mn\leq x} \Lambda(m)\Lambda(n) = 2x\log x + O(x). \tag{90}$$

This is the scale of the classical elementary input, not an estimate for the smaller covariance needed here.[3]

The cancellation in (89) can be checked locally at a zero, without an estimate for the whole zero sum. If $\zeta(s) = (s-\rho)^m g(s)$ with $g(\rho) \neq 0$, then

$$\frac{\zeta''(s)}{\zeta(s)} = \frac{m(m-1)}{(s-\rho)^2} + \frac{2m}{s-\rho}\frac{g'(s)}{g(s)} + \frac{g''(s)}{g(s)}. \tag{91}$$

For a simple zero, the double poles in $-L'$ and L^2 cancel. A simple pole may remain; it is not legitimate to say that the entire zero contribution disappears. For a multiple zero, the displayed double pole remains. A contribution of size x^β times a fixed power of $\log x$, with $\beta < 1$, is compatible with an $O(x)$ remainder. The local pole calculation therefore supplies no exclusion of a fixed zero with $1/2 < \beta < 1$.

Proposition 8.1 (A positive-density test of the proposed inference)**.** *For $t \geq 1$, let*

$$f(t) = 1 - t^{-1/4}, \qquad \widetilde{\psi}(x) = \int_1^x f(t)\,dt = x - \frac{4}{3}x^{3/4} + \frac{1}{3}.$$

[3] Atle Selberg, *An Elementary Proof of the Prime-Number Theorem*, Annals of Mathematics 50 (1949), 305–313, especially the symmetry formula in Section 2. A scan is available at https://www.math.lsu.edu/~mahlburg/teaching/handouts/2014-7230/Selberg-ElemPNT1949.pdf.

Then $f \geq 0$ and $\widetilde{\psi}(x)/x \to 1$. Nevertheless the two quantities

$$\widetilde{B}(x) = \int_1^x f(t) \log t \, dt + \iint_{\substack{u,v \geq 1 \\ uv \leq x}} f(u) f(v) \, du \, dv,$$
$$\widetilde{C}(X) = \int_1^X \left(\frac{\widetilde{\psi}(y)}{y} - 1 \right) (f(y) - 1) \, dy$$

satisfy the exact formulas

$$\widetilde{B}(x) = 2x \log x - 10x + \frac{32}{3} x^{3/4} - \frac{2}{3}, \tag{92}$$
$$\widetilde{C}(X) = \frac{8}{3} \sqrt{X} + \frac{4}{3X^{1/4}} - 4. \tag{93}$$

In particular, the analogue of (90) holds while the covariance grows as $X^{1/2}$.

Proof. The two terms defining $\widetilde{B}$ are respectively

$$x \log x - x - \frac{4}{3} x^{3/4} \log x + \frac{16}{9} x^{3/4} - \frac{7}{9},$$
$$x \log x - 9x + \frac{4}{3} x^{3/4} \log x + \frac{80}{9} x^{3/4} + \frac{1}{9}.$$

The $x^{3/4} \log x$ terms cancel, giving (92). Also

$$\left(\frac{\widetilde{\psi}(y)}{y} - 1 \right) (f(y) - 1) = \frac{4}{3} y^{-1/2} - \frac{1}{3} y^{-5/4},$$

whose integral is (93). □

Scope of the countermodel. This calculation refutes the implication from nonnegativity, PNT, and the displayed symmetry asymptotic alone to the proposed covariance bound. The density $f(t)\,dt$ is not the prime-power measure $d\psi(t)$. It does not retain the integer divisor identities, the Möbius inverse, or the complete Euler product. Consequently it does not refute a future argument that uses those additional arithmetic properties. It shows exactly why replacing them by the weakened inputs loses essential information.

8.2 Completed modulus comparisons and their exact threshold

Use the completed entire function

$$\xi_\zeta(s) = \tfrac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s), \qquad \xi_\zeta(s) = \xi_\zeta(1-s),$$

and let θ be the supremum of the real parts of its zeros. Horizontal modulus monotonicity of ξ_ζ in an open zero-free right half-plane is a proved result. Extending that monotonicity to $\Re s > 1/2$ is itself an RH reformulation.[4] The shifted comparison below makes the threshold explicit.

Proposition 8.2 (Sharp threshold for the shifted comparison)**.** *For a fixed $h > 0$, the inequality*

$$|\xi_\zeta(s+h)| > |\xi_\zeta(s-h)| \qquad (\Re s > 1/2) \tag{94}$$

holds if and only if $\theta \leq 1/2 + h$.

[4] Jonathan Sondow and Cristian Dumitrescu, *A Monotonicity Property of Riemann's Xi Function and a Reformulation of the Riemann Hypothesis*, Theorem 1 and Corollary 1: https://arxiv.org/pdf/1005.1104.

Proof. If a zero ρ has $\Re\rho > 1/2 + h$, take $s = \rho - h$. Then the left side of (94) is zero, so the strict inequality fails.

Conversely, suppose all zeros satisfy $|\Re\rho - 1/2| \le h$, using functional-equation symmetry for the left boundary. Group the even Hadamard product of $\xi_\zeta(1/2 + z)$ into conjugate reflected groups; there is no nonconstant exponential factor in this even product. For a reflected pair at $1/2 \pm a + i\gamma$, put $x = \Re s - 1/2 > 0$ and $y = \Im s - \gamma$. The difference of the squared moduli of its two polynomial factors is

$$\begin{aligned}\left|((x+h)+iy)^2 - a^2\right|^2 - \left|((x-h)+iy)^2 - a^2\right|^2 \\ = 8xh(x^2 + h^2 + y^2 - a^2) > 0,\end{aligned} \tag{95}$$

because $h \ge |a|$. For a zero on the symmetry line, the difference for its single linear factor is $|(x+h)+iy|^2 - |(x-h)+iy|^2 = 4xh > 0$. Multiplying the inequalities in the convergent grouped product proves (94); a zero on its right-hand side causes no difficulty. □

Thus a known zero-free boundary $\theta_0 > 1/2$ supplies this comparison for $h \ge \theta_0 - 1/2$. It gives no iteration toward $h = 0$. This is consistent with the classical shifted sum and difference results: the unconditional $|h| \ge 1/2$ range and the RH-conditional extension to every nonzero h are explicitly distinguished in Lagarias's treatment.[5]

The same obstruction appears in the logarithmic derivative. Away from zeros, put $U(s) = \Re(\xi_\zeta'(s)/\xi_\zeta(s))$. All archimedean and polar terms must be retained:

$$U(s) = \Re\left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}\right) - \frac{1}{2}\log\pi. \tag{96}$$

The functional equation gives $U(1 - \sigma + it) = -U(\sigma + it)$, and therefore $U(1/2 + it) = 0$ where defined. This boundary relation does not make U harmonic on a half-plane containing an unknown zero. For the pair above, its local pole contribution is

$$\frac{x-a}{(x-a)^2 + y^2} + \frac{x+a}{(x+a)^2 + y^2} = \frac{2x(x^2 + y^2 - a^2)}{((x-a)^2 + y^2)((x+a)^2 + y^2)}. \tag{97}$$

It is negative when $x > 0$ and $x^2 + y^2 < a^2$. Near an off-line zero the rational and gamma terms in (96) are holomorphic and do not cancel this pole. A maximum-principle argument that omits the interior pole has already assumed the required zero exclusion. The prime series for $-\zeta'/\zeta$ is absolutely justified only in $\Re s > 1$; its termwise use farther left needs a separate argument.

8.3 The actual forcing: decaying correlations and dense shifts

Retain the corrected d, b introduced above, extended by zero below 1, so that

$$(Ad)(x) = \sum_{n \le x} \frac{d(x/n)}{n} = b(x), \qquad b(x) = O\left(\frac{\log(2x)}{x}\right).$$

Finite Möbius inversion is exact. For a cutoff X, set

$$d_X(x) = \sum_{n \le X} \frac{\mu(n)}{n} b(x/n). \tag{98}$$

[5] Jeffrey C. Lagarias, *Zero Spacing Distributions for Differenced L-Functions*, especially Section 2: https://arxiv.org/pdf/math/0601653. The threshold proposition above is a direct canonical-product calculation for the present comparison, not an assertion of priority.

It agrees with $d(x)$ when $1 \le x \le X$. Fix $0 < \varepsilon < 1$, and write

$$a = \frac{1-\varepsilon}{2}, \quad \sigma = \frac{1+\varepsilon}{2}, \quad H_\varepsilon = L^2((0,\infty), x^{-\varepsilon}\,dx), \quad g(u) = e^{au} b(e^u).$$

Here $g(u) = 0$ for $u < 0$. The actual forcing gives $|g(u)| \ll (1+u)e^{-\sigma u}$ for $u \ge 0$. Equation (98) becomes

$$e^{au} d_X(e^u) = \sum_{n \le X} \mu(n) n^{-\sigma} g(u - \log n).$$

With the real, even autocorrelation $K(v) = \int_{\mathbb{R}} g(u)g(u+v)\,du$, this proves

$$\|d_X\|_{H_\varepsilon}^2 = \sum_{m,n \le X} \frac{\mu(m)\mu(n)}{(mn)^\sigma} K\!\left(\log \frac{n}{m}\right), \qquad |K(v)| \ll_\varepsilon (1+|v|)e^{-\sigma|v|}. \tag{99}$$

Why a diagonal estimate fails. The forcing is nonzero, so $K(0) = \|g\|_2^2 > 0$. Continuity gives a fixed $\delta > 0$ such that $K(v) \ge K(0)/2$ for $|v| \le \delta$. For all $m, n \in [N, e^\delta N]$, their Gram entries therefore have this positive lower bound. Taking coefficients equal to one on this block gives a quadratic form of order N^2, whereas the sum of the squared coefficients is only of order N. Thus there is no uniform inequality

$$\left\| \sum_n c_n g(\,\cdot - \log n) \right\|_2^2 \le C \sum_n |c_n|^2.$$

This block test refutes a proposed operator estimate; it does not give a lower bound for the actual signed Möbius sum. Although $\sum_n \mu(n)^2 n^{-2\sigma} < \infty$, that fact cannot be combined with the false diagonal estimate to conclude convergence.

Proposition 8.3 (The valid Schur bound and its density loss)**.** *For the actual g and every finite scalar sequence,*

$$\left\| \sum_n c_n g(\,\cdot - \log n) \right\|_2^2 \ll_\varepsilon \sum_n n|c_n|^2. \tag{100}$$

Consequently,

$$\int_1^X |d(x)|^2 x^{-\varepsilon}\,dx \le \|d_X\|_{H_\varepsilon}^2 \ll_\varepsilon \sum_{n \le X} \mu(n)^2 n^{-\varepsilon} \ll_\varepsilon X^{1-\varepsilon}. \tag{101}$$

Proof. Apply Schur's test to $|K(\log(n/m))|/\sqrt{mn}$. For $2^j m \le n < 2^{j+1}m$, the row sum is $O_\varepsilon((1+j)2^{j(1/2-\sigma)})$, which is summable because $\sigma > 1/2$. The ranges $n < m$ are summable as well. The matrix is symmetric, so its row and column sums are uniformly bounded. Replacing the input coefficient by $\sqrt{n}\,c_n$ proves (100). Substituting $c_n = \mu(n)n^{-\sigma}$ proves (101). □

The factor n is the cost of packing order n logarithmic shifts into a fixed-length interval. The calculation controls separated dilations, but it does not supply the required cancellation inside the dense blocks. Positivity of the forcing is also unavailable: for the explicit b,

$$b(2) = \tfrac{1}{4}\log 6 - \tfrac{1}{2} < 0, \qquad b(5/2) = \tfrac{1}{5}\log 30 - \tfrac{1}{2} > 0.$$

The bound (101) is unconditional and retains a positive power of X; it is not the desired inverse estimate.

8.4 A compact check on the weighted dyadic shortcut

For $r(x) = \psi(x)/x - 1$ and $Tf(x) = f(2x)$, one has $D = (T - I)r$. On H_ε, for $0 \le \varepsilon < 1$,

$$\|T\| = q_\varepsilon = 2^{(\varepsilon-1)/2} < 1, \qquad (1 - q_\varepsilon)\|r\|_{H_\varepsilon} \le \|D\|_{H_\varepsilon} \le (1 + q_\varepsilon)\|r\|_{H_\varepsilon}.$$

If only D is initially known to have finite norm, the Neumann series $-\sum_{k\ge0} T^k D$ converges there. Its partial sums are $r(x) - r(2^K x)$, so PNT identifies its limit with r. This proves equivalence of the finite-norm conditions, not their finiteness.

Under $x = e^u$, the requested measure dx becomes $e^u\,du$. Replacing it by $du = dx/x$ removes the difficult weight. In fact a classical PNT error bound already makes $\int_1^\infty D(x)^2\,dx/x$ finite, without proving the required subpower bound for $\int_1^X D(x)^2\,dx$.[6] No arithmetic estimate is gained by confusing these two square functions.

Status of these attempts. The identities, threshold, countermodel, and Schur bound above are proved within their stated scopes. The additional signed arithmetic cancellation, the required inverse estimate, and the all-character extension remain unproved. None of these calculations asserts RH or GRH.

9 What remains, and what would complete the argument

The successful repairs are component results. They do not identify the full arithmetic trace with the spectrum retained by the chosen completion, and the subsequent arithmetic calculations do not prove the necessary cancellation. This section records the remaining targets in one place.

9.1 The rational arithmetic target

For the definitions in Sections 5–7, each of the following assertions, with its indicated quantifiers, is equivalent to RH:

1. **Subpower prime energy:** $\mathcal{J}_d(X) \ll_\varepsilon X^\varepsilon$ for every $\varepsilon > 0$.
2. **One-sided covariance control:** $C(X) \le K_\varepsilon X^\varepsilon$ for every $\varepsilon > 0$ and all sufficiently large X, where $C(X) = \int_{(1,X]} r(y-)(d\psi(y) - dy)$ and K_ε is fixed.
3. **Subpower mollifier norm:** $\mathcal{N}_N \ll_\varepsilon N^\varepsilon$ for every $\varepsilon > 0$.

The equivalences are proved in Theorems 6.1 and 5.1 and the covariance criterion in Section 6. They give several exact formulations of the same unresolved zero-location problem, not a chain of independently established improvements.

The forcing b has been computed explicitly and lies in $L^2(1,\infty)$. Its exact inverse is

$$d(x) = \sum_{n\le x} \frac{\mu(n)}{n} b(x/n).$$

What is missing is an estimate for the *actual signed sum* strong enough to prove the first target. Absolute summation gives only $O(X)$ for its energy. The logarithmic diagonal and the gcd decomposition do

[6] For a concrete unconditional bound of this type, see Daniel R. Johnston and Andrew Yang, *Some explicit estimates for the error term in the prime number theorem*: https://arxiv.org/abs/2204.01980. Its exponential decay in $\sqrt{\log x}$ is integrable after the substitution $u = \log x$ in the logarithmic measure.

not bound the off-diagonal contribution. In the balanced primitive range the common divisor is one, and the cutoff kernel is not multiplicative, so common-divisor averaging does not remove the remaining Möbius signs.

The prime-pair kernel gives the same limitation in different variables: its continuous density main term cancels exactly, but positive and negative weighted contributions both have size proportional to X. No subpower bound for their difference has been proved. A usable new argument must control that arithmetic discrepancy rather than infer it from the positivity of the total square or from cancellation of the density term.

Accepting the reported external zero-free boundary $\Re\rho \le 7/8$ gives the conditional consequence

$$\mathcal{J}_d(X) \ll X^{3/4}$$

by Proposition 6.6. The exponent is fixed; it does not imply the estimate for every $\varepsilon > 0$. The smoothing, shifted-ξ, and Gram calculations recorded here supply no iteration that lowers 7/8 to 1/2.

9.2 The earlier analytic comparison is also unresolved

For the rational unweighted Paley–Wiener model, Theorem 2.7 identifies the closed source image as $B_\zeta H^2$. It is therefore enough, and necessary, to construct allowed theta profiles converging to e^{-x} in $L^2(0,\infty)$. The model-space distance formula in Corollary 2.8 quantifies why an off-line zero prevents this approximation. No such approximating sequence, or proof that its distance is zero, has been supplied.

The other sufficient analytic targets retained from the earlier supplement are local boundedness of the renormalized Euler products in Theorem 3.5, the subexponential support-loss bound for the Weil form in Corollary 4.3, and the weighted approximation in (51) together with the stated spectral hypotheses. Each needs an additional estimate. Finite moment cancellation, local cutoff convergence, and the positive archimedean contribution do not supply it.

9.3 What the all-character GRH claim would require

The rational trivial-character calculation proves an RH criterion. For a primitive Dirichlet character χ, set

$$\psi_\chi(x) = \sum_{n\le x} \Lambda(n)\chi(n), \qquad D_\chi(x) = \frac{\psi_\chi(2x)}{2x} - \frac{\psi_\chi(x)}{x}.$$

The corresponding all-character target is

$$\int_1^X |D_\chi(x)|^2\,dx \ll_{\chi,\varepsilon} X^\varepsilon \quad \text{for every primitive } \chi \text{ and every } \varepsilon > 0. \tag{102}$$

For qualitative GRH the constants may depend on the fixed character; uniformity in the conductor is not required for this statement. Compactly supported L^2 corrections do not change it.

Initially for $\Re s > 1$, the same transform calculation gives

$$\int_1^\infty D_\chi(x)x^{-s}\,dx = -\frac{2^{s-1}-1}{s}\frac{L'(s,\chi)}{L(s,\chi)}.$$

At a nontrivial zero ρ with $1/2 < \Re\rho < 1$, the residue is $-m_\rho(2^{\rho-1}-1)/\rho \ne 0$. Thus (102) would exclude zeros to the right of 1/2 for each character. Applying the functional equation to the complete

family, including $\overline{\chi}$, then gives Dirichlet GRH. Conversely, Dirichlet GRH gives, for fixed χ, the usual $O_\chi(\sqrt{x}\log^2(2x))$ prime-counting error and hence (102). The principal main term cancels in D_χ.

No proof of (102) has been obtained here. The original manuscript's wider number-field and Hecke-character claims would additionally require their corresponding arithmetic comparison, local terms, and all-character estimates. The zeta specialization does not establish those assertions.

Conclusion of the audit. The writeup now contains the repaired analytic components, explicit arithmetic reductions, rigorous tests of the attempted shortcuts, and precise remaining estimates. The central subpower bound is still unproved. No statement in this supplement asserts that RH or GRH has been proved.

Sources and attribution

The following sources also identify the classical inputs to the new arithmetic sections. The proofs in this supplement are supplied for the particular constructions under discussion; they are not claims of priority over the classical closure, Hardy-space, or Weil-positivity literature. Manuscript page numbers below refer to the printed pages of the uploaded PDF.

1. Alessandro Maria Masullo, *The Music of the Primes*, uploaded 940-page draft. In particular Definitions 7.3.2.1 and 11.2.1.1, Theorem 11.2.1.2, Remark 10.4.1.4, the cutoff discussion on pp. 514–516, and Theorem 13.3.1.1 with condition (13.3.1.2), p. 522.
2. *Analytic repairs and reconstruction limits*, preceding 39-page companion note, expanded 20 November 2026. The present proofs are organized as a separate supplement; the Chapter 13 qualification on the first page applies to the earlier constant-shell obstruction.
3. Arne Beurling, *A closure problem related to the Riemann zeta-function*, Proceedings of the National Academy of Sciences 41 (1955), 312–314. https://pmc.ncbi.nlm.nih.gov/articles/PMC528084/.
4. Michel Balazard and Éric Saias, *The Nyman–Beurling equivalent form for the Riemann hypothesis*, ESI Preprint 623 (1998). Theorem 0 and Sections 2–4 describe the classical Blaschke-factor closure mechanism. The present theta-source proof is an adaptation to the source space at issue, not a new general Hardy-space criterion. https://www.esi.ac.at/preprints/esi623.pdf.
5. Alain Connes and Caterina Consani, *Weil positivity and Trace formula, the archimedean place.* Equation (3), Theorem 1, and Appendix C provide the explicit-formula normalization and finite-condition criterion used for comparison. https://alainconnes.org/wp-content/uploads/Selecta.pdf.
6. Ralf Meyer, *On a representation of the idele class group related to primes and zeros of L-functions.* https://arxiv.org/abs/math/0311468.
7. Ethan Simpson Lee and Paweł Nosal, *Sharper bounds for the error in the prime number theorem assuming the Riemann Hypothesis.* https://arxiv.org/abs/2312.05628.
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